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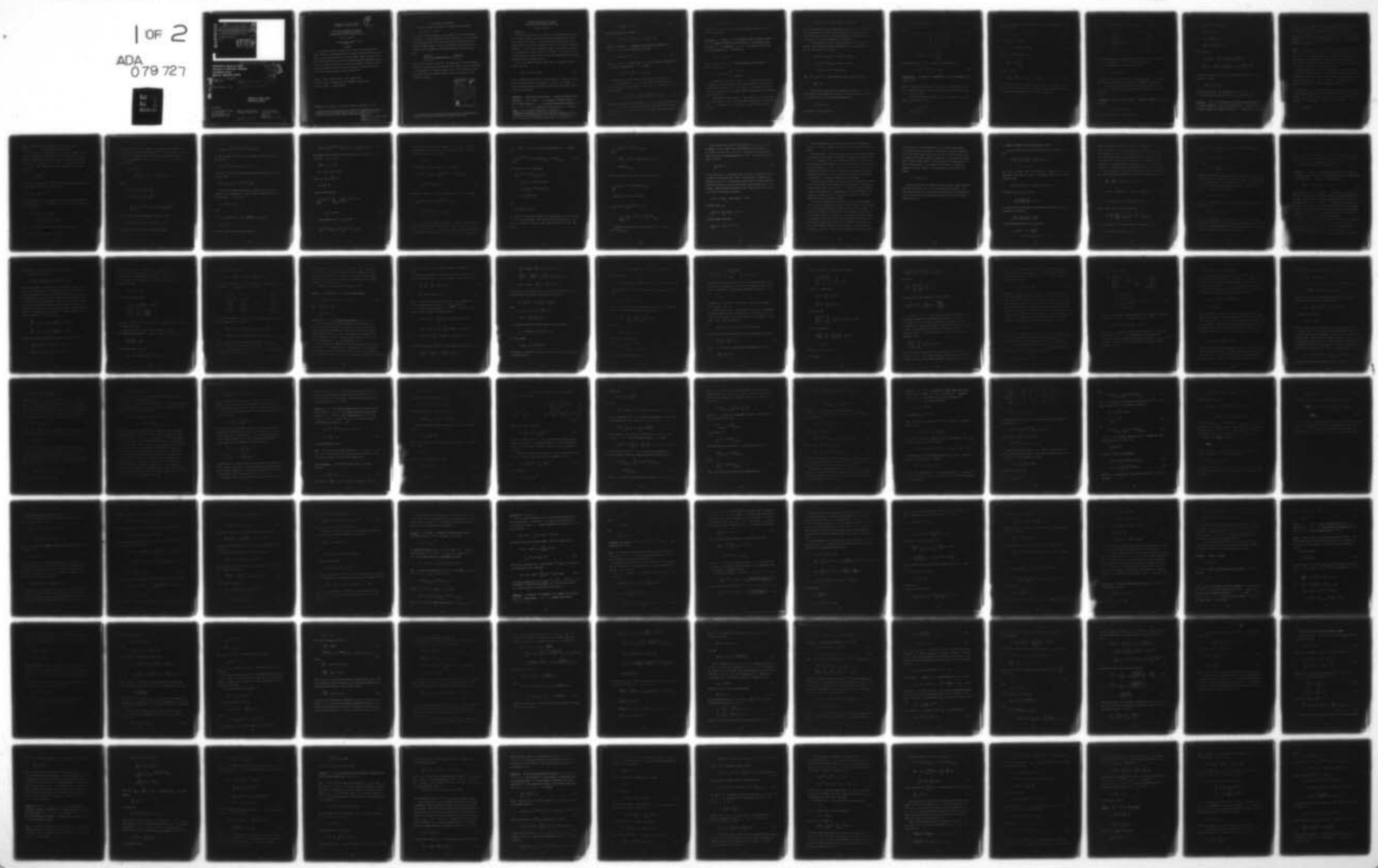

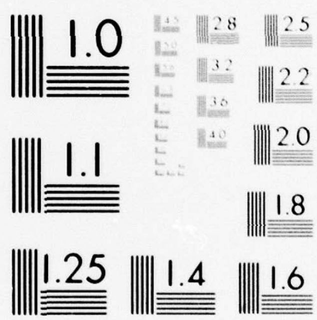
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(6) CLOSED-LOOP EIGENVALUE SPECIFICATION FOR
INFINITE DIMENSIONAL SYSTEMS: AUGMENTED
AND DEFICIENT HYPERBOLIC CASES.

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CLOSED-LOOP EIGENVALUE SPECIFICATION FOR INFINITE DIMENSIONAL SYSTEMS: AUGMENTED AND DEFICIENT HYPERBOLIC CASES

David L. Russell*

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ABSTRACT

We introduce a method for reduction of a class of distributed parameter control systems to a corresponding control canonical form which has the same intermediate "control Jordan form" as the original system. Using this reduction together with certain results concerning the completeness and independence properties of complex exponentials we are able to study the problem of eigenvalue specification (pole placement for certain classes of hyperbolic systems which are more general than those discussed in the earlier paper [20].

AMS(MOS) Subject Classifications: 93B10, 42A64, 42A08

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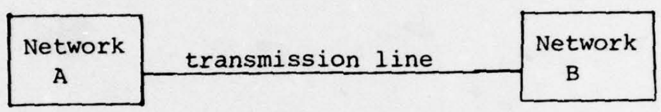
SIGNIFICANCE AND EXPLANATION

This report discusses the design of a closed loop control system

$$\dot{x} = Ax + bu, \quad u = (x,f)$$

for a certain class of operators A, input elements b, and linear feedback functionals f. The design objective is that of selecting f in such a manner that the spectrum of the closed loop control system coincides with a given sequence of complex numbers. The class of systems considered includes, for example, that of transmission line coupled to electronic networks at both ends,



which occurs very widely in practice. The significance of spectral assignment results lies in the fact that the spectrum of the closed loop system corresponds, to a large degree, to its dynamic behavior and our results show that, within well defined limits, the dynamic behavior of the controlled system can be arbitrarily assigned.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CLOSED-LOOP EIGENVALUE SPECIFICATION
FOR INFINITE DIMENSIONAL SYSTEMS:
AUGMENTED AND DEFICIENT HYPERBOLIC CASES

David L. Russell*

1. Introduction

Our purpose in this article is to discuss a certain approach to the question of control canonical structure and eigenvalue specification for certain infinite dimensional systems. In the present work we consider only certain systems which are essentially of hyperbolic type (or, in the case of functional equations, neutral type), but nevertheless a wider class than that considered in [20], which was our first approach to this subject. In later work we hope to consider parabolic systems (see [21], however, for some preliminary results), retarded functional equation systems, etc.

Let H be a complex, separable Hilbert space. We consider a control system of the form

$$\dot{x} = Ax + bu, \quad x \in H, \quad u \text{ scalar}, \tag{1.1}$$

where A generates a strongly continuous semi-group $S(t)$ (actually, a group for all systems considered in this paper) on H and b is an element of H or else an admissible control distribution element as we shall define shortly. We first of all introduce

Assumption A The spectrum of the operator A consists of distinct complex eigenvalues $\{\lambda_j \mid j \in J\}$, where J is an appropriate countable index set, each of finite multiplicity μ_j. The associated generalized eigenvectors $\phi_{j,\ell}$, $j \in J$, $\ell = 1,2,\ldots,\mu_j$, form a uniform basis (Riesz basis) for H. Thus each element $x \in H$ has a unique representation, convergent in H,

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$$x = \sum_{j\in J} \sum_{\ell=1}^{\mu_j} x_{j,\ell}\, \varphi_{j,\ell} \tag{1.2}$$

and, for certain positive constants d, D

$$d^{-2} \| x \|_H^2 \leq \sum_{j\in J} \sum_{\ell=1}^{\mu_j} | x_{j,\ell} |^2 \leq D^2 \| x \|_H^2 , \quad x \in H \tag{1.3}$$

Letting x_j denote the μ_j dimensional column vector with components $x_{j,\ell}$, $\ell = 1,2,\ldots,\mu_j$, (1.1) can be represented in the form

$$\dot{x}_j = A_j x_j + b_j u, \quad j \in J, \tag{1.4}$$

where A_j is a $\mu_j \times \mu_j$ matrix having λ_j as (sole) eigenvalue with multiplicity μ_j. Further, we assume that for $j \in J$, the matrix

$$C_j = (A_j^{\mu_j - 1} b_j,\ A_j^{\mu_j - 2} b_j, \ldots,\ A_j b_j, b_j) \tag{1.5}$$

is nonsingular--a condition necessary for approximate controllability [25].

The nonsingularity of C_j in (1.5) implies that the exhibited column vectors $A_j^{\ell} b_j$ are linearly independent. Hence, with I_{μ_j} denoting the $\mu_j \times \mu_j$ identity matrix

$$(\lambda_j I_{\mu_j} - A_j)^{\mu_j - 1} b_j = \sum_{\ell=0}^{\mu_j - 1} (-1)^{\ell} \binom{\mu_j - 1}{\ell} \lambda_j^{\mu_j - 1 - \ell} A_j^{\ell} b_j \neq 0 \tag{1.6}$$

and we conclude that the nilpotent matrix $(\lambda_j I_{\mu_j} - A_j)$ has index of nilpotency μ_j.

In (1.4) the μ_j dimensional vectors b_j are formed from the expansion coefficients $b_{j,\ell}$, $\ell = 1,2,\ldots,\mu_j$, $j \in J$, of the "control distribution element" b. If $b \in H$ then

$$b = \sum_{j\in J} \sum_{\ell=1}^{\mu_j} b_{j,\ell}\, \varphi_{j,\ell},$$

convergent in H. Sometimes, as in the case of boundary controls, this is too stringent a requirement.

Definition 1.1 The element b is an admissible control distribution element if it is a (possibly unbounded) linear functional on H whose domain includes the domain of A^*. The components, $b_{j,\ell}$ of b with respect to the basis of generalized eigenvectors $\varphi_{j,\ell}$ are defined by

$$b_{j,\ell} = \langle \psi_{j,\ell}, b\rangle \tag{1.7}$$

where $\{\psi_{j,\ell} \mid j\in J,\ \ell = 1,2,\dots,\mu_j\}$ are the elements of the dual basis of H relative to the $\varphi_{j,}$, i.e.,

$$(\varphi_{j,\ell}, \psi_{\hat{j},\hat{\ell}})_H = \begin{cases} 1 & j=\hat{j},\ \ell=\hat{\ell} \\ 0 & \text{otherwise}. \end{cases}$$

The symbol $\langle \psi, b\rangle$, where $\psi\in H$ and b is a linear functional defined on some domain in H, denotes the value of the linear functional b at the point ψ, assumed to be in the domain of b. It is linear in ψ, antilinear in b, and reduces to $(\psi,b)_H$ when b is a bounded linear functional which may be identified as an element of H.

For admissible control elements b the solution of (1.1) corresponding to a control function u which is locally square integrable and an initial state $x(0) = x_0$ H defined, for $t \geq 0$, as the unique $x_u(t)\in H$ such that, for each $\psi\in D(A^*)$,

$$(\psi, x_u(t))_H = (\psi, S(t)x_0)_H + \int_0^t \langle S(t-\tau)^*\psi, b\rangle u(\tau)\, d\tau \tag{1.8}$$

where, as we recall, $S(t)$ is the semigroup generated by A in H. It remains to be shown, and we will do this a little later in this section, that this is a viable definition of $x_u(t)$. A particular admissible control element which does not lie in H plays an important role in Sections 5,6.

It is important to be able to recognize the form of b in concrete examples. This is best done from the computation, for ψ in the demain of A^* and x_0 in the domain of A, T fixed,

$$\frac{d}{dt}(S(T-t)^*\psi, x_u(t))_H = (-A^*S(T-t)^*\psi, S(t)x_0)_H$$

$$+ (S(T-t)^*\psi, AS(t)x_0)_H + \langle S(T-t)^*\psi, b\rangle u(t)$$

$$= \langle S(T-t)^*\psi, b\rangle u(t) = \langle w(t), b\rangle u(t),$$

where $w(t) = S(T-t)^*\psi$ is a differentiable solution of the adjoint system

$$\frac{dw}{dt} = -A^*w .$$

We return now to the set of finite dimensional systems (1.4). From the familiar theory of the Jordan form of a matrix, we know that there are nonsingular $\mu_j \times \mu_j$ matrices P_j such that, with the transformation

$$x_j = P_j\xi_j, \quad j\in J, \tag{1.9}$$

the systems (1.4) are carried into

$$\dot{\xi}_j = \Lambda_j \xi_j + \beta_j u, \; j \in J, \tag{1.10}$$

with

$$\Lambda_j = \lambda_j I\mu_j + N_j = \begin{pmatrix} \lambda_j & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_j & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_j & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_j \end{pmatrix} \tag{1.11}$$

$$\Lambda_j = P_j^{-1} A_j P_j, \; \beta_j = P_j^{-1} b_j = \begin{pmatrix} \beta_j^1 \\ \vdots \\ \beta_j^{\mu_j} \end{pmatrix}$$

The nonsingularity of C_j in (1.5) implies the same property for

$$\tilde{C}_j = (\Lambda_j^{\mu_j - 1} \beta_j, \Lambda_j^{\mu_j - 2} \beta_j, \dots, \Lambda_j \beta_j, \beta_j) \tag{1.12}$$

<u>Proposition 1.2</u> <u>The matrix</u> (1.12) <u>equivalently</u> (1.15)) <u>is nonsingular if and only if</u> $\beta_j^{\mu_j} \neq 0$.

<u>Proof</u> The "only if" part is easy; for if β_j, μ_j (the last component of the μ_j -- dimensional vector β_j) is zero, it is easy to verify that the last row of $\tilde{C}_j$ in (1.12) is zero.

To show the sufficiency we suppose $\beta_j, \mu_j \neq 0$ and define the $\mu_j \times \mu_j$ matrix B_j with entries $(B_j)_k^\ell$:

$$(B_j)_k^\ell = \begin{cases} \beta_j^{\mu_j - (k-\ell)} & , \; k \geq \ell \\ 0, \; k < \ell \end{cases}$$

Then B_j is triangular with diagonal entries all equal to $\beta_j^{u_j}$; hence it is non-singular. Setting

$$\xi_j = B_j \eta_j \tag{1.13}$$

we obtain, in place of (1.10),

$$\dot{\eta}_j = B_j^{-1} \Lambda_j B_j \eta_j + B_j^{-1} \beta_j \; u. \tag{1.14}$$

Since β_j is the last column of B_j,

$$B_j^{-1} \beta_j = e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{1.15}$$

and since (cf. (1.11))

$$B_j^{-1} \Lambda_j B_j = B_j^{-1} (\lambda_j I \mu_j + N_j) B_j = \lambda_j I \mu_j + B_j^{-1} N_j B_j = \lambda_j I \mu_j + N_j, \tag{1.16}$$

the last equality in (1.16) following from the fact that $N_j B_j$ has for its i-th column ($i = 2,3,\dots, \mu_j$) the (i-1)st column of B_j, due to the special structure of B_j. Under this transformation $\tilde{C}_j$ of (1.12) is carried into

$$\hat{C}_j = (\Lambda_j^{\mu_j-1} e_j, \Lambda_j^{\mu_j-2} e_j, \ldots, \Lambda_j e_j, e_j)$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \binom{\mu_j-1}{1}\lambda & 1 & \cdots & 0 & 0 \\ \binom{\mu_j-1}{2}\lambda^2 & \binom{\mu_j-2}{1}\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda^{\mu_j-1} & \lambda^{\mu_j-2} & \cdots & \lambda & 1 \end{pmatrix} \tag{1.17}$$

a lower triangular matrix with diagonal entries equal to 1. Thus $\hat{C}_j$ and hence $\tilde{C}_j$, C_j are nonsingular if $\beta_j^{\mu_j} \neq 0$ and the proof is complete

The transformation (1.13) carries (1.10) into

$$\dot{\eta}_j = \Lambda_j \eta_j + e_j u, \quad j \epsilon J, \tag{1.18}$$

This form of our system will be called the <u>control Jordan form</u>. It is a standard form to which various systems may be reduced for comparison with each other.

The largest part of this paper treats the simple eigenvalue case where $\mu_j = 1$, $j \epsilon J$. In Section 7 we consider the multiple eigenvalue case. There we will need

<u>Assumption B</u> <u>There is a positive integer</u> μ <u>and positive numbers</u> M_0, M_1, M_2, M_3 <u>such that</u>

$$\mu_j (= \text{mult. } \lambda_j) \leq \mu, \quad j \epsilon J, \tag{1.19}$$

(Statement of Assumption B continued next page.)

$$\|P_j\| \text{ (cf. (1.9))} \leq M_0,\ j\in J \tag{1.20}$$

$$\|P_j^{-1}\| \leq M_1,\ j\in J, \tag{1.21}$$

$$\|B_j\| \leq M_2\beta_j^{\mu_j} \neq 0,\ j\in J \tag{1.22}$$

$$\|B_j^{-1}\| \leq M_3 \left(\beta_j^{\mu_j}\right)^{-1},\ j\in J. \tag{1.23}$$

Using the formula

$$\Lambda_j^{-1} = \left(\beta_j^{\mu_j} I\mu_j + \tilde{N}_j\right)^{-1} = \left(\beta_j^{\mu_j}\right)^{-1}\left(I\mu_j + \left(\beta_j^{\mu_j}\right)^{-1}\tilde{N}_j\right)$$

$$= \left(\beta_j^{\mu_j}\right)^{-1}\left(I\mu_j - \left(\beta_j^{\mu_j}\right)^{-1}\tilde{N}_j + \left(\beta_j^{\mu_j}\right)^{-2}\tilde{N}_j^2 + \ldots + (-1)^{\mu_j-1}\left(\beta_j^{\mu_j}\right)^{-\mu_j+1}\tilde{N}_j^{\mu_j-1}\right)$$

it is easy to see that (1.22), (1.23) are satisfied if the first $\mu_j - 1$ components of β_j satisfy

$$m|\beta_j^{\mu_j}| \leq |\beta_j^{\ell}| \leq M|\beta_j^{\mu_j}|$$

for some positive numbers m, M independent of $j\in J$, $= 1,2,\ldots,\mu_{j-1}$.

We are now ready to study the solution $x_u(t)$ defined in (1.8).

<u>Theorem 1.3</u> <u>Let A be the generator of a strongly continuous semigroup in H for which Assumptions A and B are satisfied and let b be an admissible control distribution element with bounded coefficients</u> (1.7). <u>Suppose that the eigenvalues</u>

$\lambda_k \equiv \rho_k + i\sigma_k$ of A possess the following "parabolic density" property: there are positive numbers M and G such that, whenever $R - r \geq G$, at most $M(R - r)$ of the numbers $\sqrt{|\rho_j| + i\,\sigma_j}$, $j \in J$, lie in the region $r \leq z \leq R$ of the complex plane. Then for locally square integrable u, (1.8) defines a unique element $x_u(t)$ H for $t \geq 0$ and the function $x_u : [0,\infty) \to H$ is continuous with respect to the topology of H.

Remarks The assumption that A generates a strongly continuous semigroup already guarantees that there is some real number ρ such that $\rho_j = \mathrm{Re}(\lambda_j) \leq \rho$, $j \in J$ (see, e.g., [6]). The parabolic density requirement is not unduly restrictive, being satisfied by the eigenvalues of generating operators associated with most hyperbolic and parabolic systems in a single space dimension. For an example of an operator for which it is not satisfied, see [16].

The requirements (1.22),(1.23) of Assumption B are not actually necessary to the proof of Theorem 1.3 but will be retained here to keep the proof simple. The interested reader will easily be able to modify the proof to obtain a similar result in cases where, e.g., $\beta_j^{\mu_j} = 0$ for some j. Note that there is no assumption that the $\beta_j^{\mu_j}$ are uniformly bounded away from zero.

Proof of Theorem 1.3 To facilitate the proof we assume that the index set J has been completely ordered by some order relation $\blacktriangleleft$ and has a unique minimal element. Thus for any $j \in J$ there is a finite integer, which we shall call $|j|$, such that $|j|$ is the number of elements in $J \blacktriangleleft j$. Obviously

$$\sum_{j \in J} \frac{1}{|j|^2} < \infty .$$

We will further assume that the indexing is done in such a way that

$\left|\sqrt{|\rho_j|} + i\sigma_j\right|$ is nondecreasing with respect to $|j|$.

The essential step is to show that $x_u(t)$, defined for $\psi \epsilon \mathcal{D}(A^*)$ by (1.8), extends to a bounded linear functional on H. The requirements set forth in Assumption B allow the work to be carried out in the context of the operation $\hat{A}$ defined on ℓ^2 (indexed by $j \epsilon J$, $\ell = 1,2,\ldots, \mu_j$) by means of the blocks $\Lambda_j = \lambda_j I\mu_j + N_j$. Rather than using the change of variable (1.13), however, we will use

$$\xi_j = \left(\beta_j^{\mu_j}\right)^{-1} \beta_j \eta_j \quad . \tag{1.24}$$

From (1.22), (1.23) the matrices $\left(\beta_j^{\mu_j}\right)^{-1} B_j$ are uniformly bounded with uniformly bounded inverses. Thus we have in place of (1.18)

$$\dot{\eta}_j = \Lambda_j \eta_j + \beta_j^{\mu_j} e_j u, \; j \epsilon J, \tag{1.25}$$

as the description of our control system in ℓ^2. Let the semigroup generated by $\hat{A} = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots)$ in ℓ^2 be denoted by $\hat{S}(t)$. Then the defining equation for $\eta_u(t)$ is (cf. (1.8))

$$(\hat{\psi}, \eta_u(t))_{\ell^2} = (\hat{\psi}, \hat{S}(t)\eta_0)_{\ell^2}$$

$$+ \int_0^t (\hat{S}(t - \tau)^* \hat{\psi}, \hat{b})_{\ell^2} \overline{u(\tau)} \, d\tau \tag{1.26}$$

where $\hat{b}$ is the control distribution element with block structure

$$0,\ldots 0, \beta_1^{\mu_1}, 0\ldots 0, \beta_2^{\mu_2}, \ldots, 0\ldots 0, \beta_j^{\mu_j}, \ldots$$

which may lie in ℓ^2 or be an admissible control distribution element, i.e., a linear functional defined on the domain of $\hat{A}^*$. Our hypothesis on b guarantees that the $\beta_j^{\mu_j}$ are bounded, though not necessarily bounded away from zero.

Let the components of $\hat{\psi}\epsilon\ell^2$ be $v_{j'}$, $j\epsilon J$, $\ell = 1,2,\ldots, \mu_j$. Then it is easy to see that

$$(\hat{S}(t-\tau)^*\hat{\psi}, \hat{b})_{\ell^2} =$$

$$\sum_{j\epsilon J} \overline{\beta_j^{\mu_j}} \sum_{\ell=1}^{\mu_j} v_{j,\ell}\, e^{\overline{\lambda}_j(t-\tau)} \left((t-\tau)^{\mu_j-\ell} / (\mu_j-\ell)! \right)$$

and thus

$$\left| \int_0^t (\hat{S}(t-\tau)^*\hat{\psi}, \hat{b})_{\ell^2} \overline{u(\tau)}\, d\tau \right| \le$$

$$\sup |\beta_j^{\mu_j}| \left(\sum_{j\epsilon J} \sum_{\ell-1}^{\mu_j} |v_{j,\ell}|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{j\epsilon J} \sum_{\ell=1}^{\mu} \left| \int_0^t e^{\overline{\lambda}_j(t-\tau)} ((t-\tau)^{\mu_j-\ell} / (\mu_j-\ell)!) \overline{u(\tau)} d\tau \right|^2 \right)^{\frac{1}{2}}$$

Since the $\beta_j^{\mu_j}$ are uniformly bounded, we obtain an estimate

$$\left| \int_0^t (\hat{S}(t-\ )^*\hat{\psi},\hat{b})_{\ell^2} \; u(\tau)d\tau \right| \le K(t,u) \|\hat{\psi}\|_{\ell^2}, \quad \hat{\psi}\epsilon\ell^2, \tag{1.27}$$

for some positive $K(t,u)$, if we can show that the numbers

$$S_{j,\ell}(t,u) = \int_0^t e^{\overline{\lambda}_j(t-\tau)} \left\{ (t-\tau)^{\mu_j-\ell} / (\mu_j-\ell)! \right\} \overline{u(\tau)} d\tau$$

are square summable. Since the μ_j are bounded as a result of Assumption B, the functions

$$((t-\tau)^{\mu_j-\ell} / (\mu_j-\ell)!) u(\tau),\ j \in J,\ \ell = 1,2,\ldots,\mu_j$$

constitute a finite set of locally square integrable functions and it will be enough to study

$$S_j(t,u) = S_{j,\mu_j}(t(u) = \int_0^t e^{\mu_j(t-\tau)} \overline{u(\tau)} d\tau .$$

We carry out two different estimates, depending on the location of the points λ_j in the complex plane. From the remarks preceding this proof, we may assume that $\lambda_j = \rho_j + i\sigma_j \in H_\rho$;

$$H_\rho = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq \rho\}.$$

We let

$$R_1 = \{z \in H_\rho \mid \mathrm{Re}(z) \leq -1,\ -\frac{\pi}{4} \leq \arg\left(\sqrt{|\mathrm{Re}(z)|} + i\ \mathrm{dm}(z)\right) \leq \frac{\pi}{4}\} ,$$

$$R_2 = H_\rho - R_1 .$$

For those $\lambda_j \in R_1$, using the Schwarz inequality,

$$|S_j(t,u|^2 \le \left(\int_0^t e^{\rho\, 2\sigma_j (t-\tau)} d\tau \right) \int_0^t |u(\tau)|^2 d\tau \le \frac{1}{|\rho_j|} \int_0^t |u(\tau)|^2 \, d\tau \ . \qquad (1.28)$$

The parabolic density assumption shows that there is a positive number c such that for all $\lambda_j = \rho_j + i\sigma_j \in R_1$

$$\left|\sqrt{|\rho_j|} + i\,\sigma_j\right| \ \ge c|j|,$$

$$\text{i.e.} \quad |\rho_j| + |\sigma_j|^2 \ge c^2 |j|^2 \ .$$

But in R_1, $|\sigma_j| \ge \sqrt{|\rho_j|}$, so

$$|\rho_j| \ge \frac{c^2}{2} \ |j|^2$$

and then (1.28) shows that

$$\sum_{\substack{j\in J \\ \lambda_j \in R_1}} |S_j(t,u)|^2 \le \left(\sum_{\substack{j\in J \\ \lambda_j \in R_1}} \frac{2}{c^2|j|^2} \right) \int_0^t |u(\tau)|^2 \ d\tau,$$

$$\equiv K_1 \int_0^t |u(\tau)|^2 \, d\tau \ . \qquad (1.29)$$

On the other hand, for $\lambda_j \in R_2$, we have

$$\left|\int_0^t e^{\bar{\lambda}_j (t-\tau)} \overline{u(\tau)} d\right|^2 \le e^{2\rho t} \left| \int_0^t e^{i\sigma_j (t-\tau)} u(\tau) d\tau \right|^2 \qquad (1.30)$$

Confining attention to some finite interval $0 \leq t \leq T$, let ω_j denote the integer multiple of $\frac{2\pi}{T}$ which lies closest to the real number σ_j, subject to the condition that $|\omega_j| \leq |\sigma_j|$. Then

$$\omega_j = \frac{2\pi}{T} I_j$$

for some integer I_j which depends on $j \in J$. We compute

$$\left|\int_0^t e^{i\sigma_j(t-\tau)} u(\tau)d\tau\right|^2 \leq \left| \int_0^t \left(e^{i\sigma_j(t-\tau)} - e^{i\omega_j(t-\tau)}\right) u(\tau)\,d\tau\right|^2$$

$$+ \left|\int_0^t e^{i\omega_j(t-\tau)} u(\tau)\,d\tau\right|^2 \qquad (1.31)$$

Extending u from $[0,t]$ to $[0,T]$ by setting $u(\tau) = 0$, $t < \tau \leq T$, we have

$$\left|\int_0^t e^{i\omega_j(t-\tau)} u(\tau)\,d\tau\right| = \left|\int_0^t e^{i\frac{2\pi}{T} I_j(t-\tau)} u(\tau)\,d\tau\right|$$

$$= \left|\int_0^t e^{-i\frac{2}{T} I_j\tau} (u(\tau)d\right| = \sqrt{T}\,|a_{I_j}|$$

where a_{I_j} is the I_j-th Fourier coefficient of $u \in L^2[0,T]$. From the parabolic density requirement we infer the existence of a positive integer N such that at most N of the numbers σ_j correspond to a given ω_j under the correspondence described above, provided that the $\lambda_j = \rho_j + i\sigma_j \in R_2$, since in that region

$|\sigma_j|^2 \geq \frac{1}{2}|\sqrt{|\rho_j|} + i\sigma_j|^2$ for all but finitely many values of j. It follows that

$$\sum_{\substack{j\epsilon J \\ \lambda_j \epsilon R_2}} |\int_0^t e^{i\omega_j(t-\tau)} u(\tau)d\tau|^2 \leq NT\,||u||^2_{L^2[0,T]} . = NT\,||u||^2_{L^2[0,T]} . \tag{1.32}$$

The other term in (1.31) is estimated by

$$|\int_0^t \left(e^{i\sigma_j(t-\tau)} - e^{i\omega_j(t-\tau)}\right) u(\tau)\, d\tau|^2$$

$$= |\int_{\omega_j}^{\sigma_j} \int_0^t i(t-\tau)\; e^{is(t-\tau)} u(\tau)d\tau\; ds|^2$$

$$\leq |\sigma_j - \omega_j| |\int_{\omega_j}^{\sigma_j} |\int_0^t \; e^{is(t-\tau)} \tilde{u}(\tau)d\tau|^2\; ds$$

$$= |\sigma_j - \omega_j| |\int_{\omega_j}^{\sigma_j} |\hat{u}(s)|^2\; ds$$

where

$$\hat{u}(s) = \int_0^t e^{is(t-\tau)} \tilde{u}(\tau)\; d\tau$$

is, essentially, the Fourier transform of the function equal to $\tilde{u}(\tau) = i(t-\tau)u(\tau)$ in $[0,t]$ and zero elsewhere. Then, again using the fact that at most N of the σ_j correspond to a given $\omega_j = \frac{2\pi}{T} I_j$, and the fact that $|\sigma_j - \omega_j| \leq \frac{2\pi}{T}$, we have

$$\sum_{\substack{j\in J\\ \lambda_j\in R_2}} \left|\int_0^t \left(e^{i\sigma_j(t-\tau)} - e^{i\omega_j(t-\tau)}\right) u(\tau)\, d\tau\right|^2$$

$$\leq \frac{2\pi N}{T} \int_{-\infty}^{\infty} |\hat{u}(s)|^2\, ds = \frac{2\pi N}{T} \left(2\pi \int_0^t (t-\tau)^2 |u(\tau)|^2\, d\tau\right)$$

$$\leq 4\pi^2 NT \|u\|^2_{L^2[0,t]} \quad . \tag{1.33}$$

Using (1.32) and (1.33) in (1.30) and (1.31) we finally have

$$\sum_{\substack{j\in J\\ \lambda_j\in R_2}} |s_j(t,u)|^2 \leq e^{2\rho T} NT(1+4\pi^2) \|u\|^2_{L^2[0,t]} \quad . \tag{1.34}$$

Combining (1.29) with (1.34) we have

$$\sum_{j\in J} |s_j(t,u)|^2 = \sum_{j\in J} \left|\int_0^t e^{\overline{\lambda}_j(t-\tau)} \overline{u}(\tau)\, d\tau\right|^2$$

$$\leq \left[\sum_{\substack{j\in J\\ \lambda_j\in R_1}} \frac{2}{c^2|j|^2} + e^{2\rho T} NT(1+4\pi^2)\right] \|u\|^2_{L^2[0,t]} \quad . \tag{1.35}$$

The same sort of estimate could be obtained with $u(\tau)$ replaced by $((t-\tau)^{\mu_j-\ell}/(\mu_j-\ell)!u(\tau)$.

Each of the functions $\left[(t-\tau)^{\mu_j-\ell}/(\mu_j-\ell)!\right]u(\tau)$, $\ell = 1,2,\ldots,\mu_j \leq \mu$ can be bounded in $L^2[0,t]$ norm by some constant times the $L^2[0,t]$ norm of u. As a result we see that $K(t,u)$ in (1.27) can be taken to be some positive multiple of the last expression appearing in (1.35). It is clear that, for each fixed $u \in L^2[0,t]$,

$$\lim_{t\to 0} K(t,u) = 0 \tag{1.36}$$

At this point, then, it is established that $(\hat{\psi}, \eta_u(t))$ is defined by (1.26) as a bounded linear functional for $\hat{\psi} \in \mathcal{D}(\hat{A}^*)$ which extends to a bounded linear functional on all of ℓ^2 (the foregoing estimate takes care of the second term on the right hand side of (1.26) and the first causes no problem since $\hat{S}(t)$ is bounded). Using the strong continuity of $\hat{S}(t)$ together with (1.27) it is easy to see that for fixed square integrable u,

$$\| \eta_u(t) - \hat{S}(t)\eta_0 \|_{\ell^2} \leq \text{some constant} \times K(t,u)$$

and hence, from (1.36)

$$\lim_{t\to 0} \eta_u(t) = \lim_{t\to 0} \hat{S}(t)\eta_0 = \eta_0 \text{ in } \ell^2$$

A similar argument shows that

$$\lim_{t\to\hat{t}} \eta_j(t) = \eta_u(\hat{t}) \text{ in } \ell^2,\ t \geq 0.$$

Using the boundedness and bounded invertibility of the transformation from the x to the η variable (cf. (1.19),(1.24)) we see that our theorem is proved.

That the operator $\hat{A}$ with the block structure Λ_j, $j\epsilon J$, satisfying Assumption B and with $\mathrm{Re}(\lambda_j)$ bounded above generates a semigroup of bounded operators on ℓ^2 is a consequence of the Hille-Yoshida theory [6],[10]. One might still question whether it is meaningful to speak of solutions of (1.10) since the control distribution element $(e_1, e_2, \ldots, e_j, \ldots)$ does not lie in ℓ^2. Thus one of the consequences of Theorem 1.3 is that it establishes the existence of a unique continuous solution of (1.10) for locally square integrable u, provided that the eigenvalues λ_j satisfy the parabolic density requirement.

In this paper and, hopefully, in others to follow, we accomplish the reduction of certain linear infinite dimensional control systems to an appropriate control canonical form ([3],[20[,[23]), i.e., a scalar functional equation with the same control Jordan form. We summarize, briefly, our development of this program in the remaining sections of the present paper which is devoted to augmented and deficient hyperbolic systems.

In Section 2 we provide definitions and, we believe, significant examples, of what we mean by augmented and deficient hyperbolic systems. Then we proceed to the machinery which is needed to develop the control canonical forms. In Section 3 we introduce the Hilbert spaces $H^m[a,b]$ (Sobolev spaces when $n \geq 0$) for an arbitrary integer n, positive, zero or negative and extend the theory of non-harmonic Fourier series to include uniform bases for such spaces. Then in Section 4 we study scalar linear neutral equations of finite order (the order can be negative!) and show their relationship to the spaces developed in Section 3. Section 5 is devoted to the description of these scalar linear

neutral equations in a form comparable to (1.10), with particular emphasis on estimating the control distribution coefficients. In Section 6 we show how these scalar linear neutral equations play the role of control canonical forms for augmented and deficient hyperbolic systems and, using these control canonical forms, we obtain results similar to those obtained in [20] regarding the placement of eigenvalues when u is synthesized in a system (1.1), which is of augmented or deficient hyperbolic type, by means of continuous linear state feedback

$$u = (x,f)_H, \quad f \in H.$$

The work of Sections 3-6 is carried out in the context of simple eigenvalues, where each of the blocks Λ_j in (1.11) is replaced by the scalar λ_j. The theory extends equally well to the case where finitely many of the multiplicities μ_j are greater than one. The modifications necessary to do this are reviewed briefly in Section 7.

2. Examples of Augmented and Deficient Hyperbolic Systems

In [20] we studied a class of linear hyperbolic control systems of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} w \\ v \end{pmatrix} - A(x)\begin{pmatrix} w \\ v \end{pmatrix} = g(x)u(t) \tag{2.1}$$

$$0 \le x \le 1, \quad t \ge 0$$

where w, v, u are scalar, $A(x)$ is a continuous 2×2 matrix and $g(x)$ a two dimensional vector function in $L_2^2[0,1]$. The boundary conditions were assumed to have the form

$$a_0 \ (0,t) + b_0 v(0,t) = 0, \quad a_1 w(1,t) + b_1 v(1,t) = 0, \tag{2.2}$$

for scalars a_0, b_0, a_1, b_1 such that

$$\gamma \equiv \frac{(a_1 - b_1)(a_0 + b_0)}{(a_1 + b_1)(a_0 - b_0)} \ne 0 \text{ or } \infty \ . \tag{2.3}$$

The simplicity of the development in that study arose from the fact that the eigenvalues of the operator

$$L_0\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} w \\ v \end{pmatrix} + A(x)\begin{pmatrix} w \\ v \end{pmatrix}$$

with the boundary conditions (2.2) take the form

$$\lambda_j - \frac{1}{2}\log\gamma + j\pi i + \mathcal{O}\left(\frac{1}{|j|}\right),$$

$$j \in J = \{j \mid -\infty < j < \infty\},$$

and, as a consequence (see [14], [12], [24], [19] the exponentials $e^{\lambda_j t}$ form a uniform basis for $L^2[0,2]$ (equivalently, for $L^2[-1,1]$). Ultimately, every argument of the paper was based on this fact. In [20] all eigenvalues were assumed simple but the work carries over with very little modification to the case where r groups, each consisting of n_r eigenvalues λ_j, are replaced by r multiple eigenvalues of multiplicity R_r. In Section 7 of this paper we discuss the modifications that are necessary for multiple eigenvalues.

It is not difficult ot find hyperbolic systems which do not fit this pattern. Very simple examples may be obtained by considering the second order system

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = g(x)u(t), \quad 0 \leq x \leq 1, \; t \geq 0 \tag{2.4}$$

$$a_0 w(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \quad a_1 w(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = 0 , \tag{2.5}$$

$$g \in L^2[0,1], \; |a_0|^2 + |b_0|^2 \neq 0, |a_1|^2 + |b_1|^2 \neq 0 . \tag{2.6}$$

With $v = \frac{\partial w}{\partial t}$, (2.4) has the first order form

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{\partial^2 w}{\partial x^2} \end{pmatrix} + \begin{pmatrix} 0 \\ g(x)u(t) \end{pmatrix} \equiv L \begin{pmatrix} w \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(x)u(t) \end{pmatrix} \tag{2.7}$$

If we take $b_0 = b_1 = 0$ the eigenvalues of the operator L are

$$\lambda_j = j\pi i, \; j \in J = \{j \,|\, j = \pm 1, \pm 2, \dots\} .$$

If we take $b_0 \neq 0$, $b_o \neq 0$ it may be seen that the eigenvalues of the operator L in (2.7) take the form

$$\lambda_j, \; j \in J = \{j \mid -\infty < j < \infty\} \tag{2.8}$$

with

$$\lambda_j = j\pi i + \mathcal{O}\left(\frac{1}{|j|}\right), \; -\infty < j < \infty,$$

together with an additional eigenvalue which we will designate as σ. This additional eigenvalue, σ, cannot be included in the sequence (2.8) without disturbing the indicated asymptotic relationship between the λ_j and the $j\pi i$. This is the situation if there are no multiple eigenvalues. If we take $b_o \neq 0$, $b_1 \neq 0$, $_o = 0$, $a_1 = 0$, we have eigenvalues

$$\lambda_j = j\pi i, \; -\infty < j < \infty ,$$

but $\lambda_0 = 0$ has multiplicity two. In the case of simple eigenvalues the relevant exponentials are

$$e^{\lambda_j t}, \; -\infty < j < \infty, \; e^{\sigma t} . \tag{2.9}$$

These are not independent in $L^2[0,2]$ because $e^{\lambda_j t}$, $-\infty < j < \infty$, already form a basis for that space. The set (2.9) thus consists of a basis for $L^2[-1,1]$ augmented by a further exponential function. In the case where $\lambda_0 = 0$ is an eigenvalue of multiplicity 2 the relevant functions are

$$e^{k\pi i t},\ 0 < k < \infty,\ t. \tag{2.10}$$

The first group already forms a basis for $L^2[-1,1]$ and the whole set (2.10) thus consists of a basis augmented by another function, namely t. We again say that such a system is an augmented hyperbolic system.

Here, for the record, are the formal definitions.

Definition 2.1 A system (1.1) satisfying Assumptions A and B of Section 1 is a deficient hyperbolic system if the generalized exponentials (corresponding to the multiplicities μ_j of the λ_j as eigenvalues of A)

$$\frac{t^{\mu_j - \ell}}{(\mu_j - \ell)!}\, e^{\lambda_j t},\quad j \in J,\ \ell = 1,2,\ldots,\mu_j \tag{2.11}$$

are not complete in $L^2[-a,a]$ for some $a > 0$ but become a uniform basis for that space with adjoining of finitely many additional generalized exponential functions. Such a system is an augmented hyperbolic system if the set (2.11) is not independent in $L^2[-a,a]$ for some $a > 0$ but becomes a uniform basis for that space on the removal of finitely many functions from the set. The number of functions which must be adjoined (removed) is the index of deficiency (augmentation). If the set (2.11) is a uniform basis for $L^2[-a,a]$ for some $a > 0$ the system (1.1) is said to be an exact hyperbolic system.

Systems (1.1) satisfying Assumptions A and B are approximately controllable ([5]) in time $2a$ with controls in $L^2[0,2a]$ (or $L^2[-a,a]$) if they are deficient or exact, uniquely controllable if exact. Augmented systems are not approximately controllable in time $2a$. (see [19], [22]).

We proceed now to give some further examples of augmented and deficient

systems which are likely to be of importance in applications.

We consider again the system

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} w \\ v \end{pmatrix} - A(x)\begin{pmatrix} w \\ v \end{pmatrix} = g(x)u(t) \tag{2.12}$$

with the same assumptions as before--except as regards the boundary conditions as we will explain presently. A suitable physical analog for this system is that of a transmission line. In the context of this analogy one is reminded that transmission lines do not exist by themselves--they are usually coupled to electronic sending/receiving devices at each end. Supposing these devices to be physically small so that delays in signal propagation within the devices themselves are negligible, we may represent their dynamics by finite dimensional systems of ordinary differential equations. We therefore consider a composite system consisting of (2.12) and the systems

$$\frac{dy_0}{dt} = A_0 y_0 + g_0 u(t) + G_0 \begin{pmatrix} w(0,t) \\ v(0,t) \end{pmatrix}, \quad y_0,\ b_0 \in E^{n_0}, \tag{2.13}$$

$$\frac{dy_1}{dt} = A_1 y_1 + g_1 u(t) + G_1 \begin{pmatrix} w(1,t) \\ v(1,t) \end{pmatrix}, \quad y_1,\ b_1 \in E^{n_1} \tag{2.14}$$

together with boundary conditions to be satisfied by solutions of (2.12):

$$a_0 w(0,t) + b_0 v(0,t) + c_0^* y_0(t) = 0, \quad c_0 \in E^{n_0}, \tag{2.15}$$

$$a_1 w(1,t) + b_1 v(1,t) + c_1^* y_1(t) = 0, \quad c_1 \in E^{n_1}. \tag{2.16}$$

Here A_0, A_1, G_0, G_1 are constant matrices of dimension $n_0 \times n_0$, $n_1 \times n_1$, $n_0 \times 2$, $n_1 \times 2$, the coefficients a_0, b_0, a_1, b_1, satisfy (2.3) and * denotes the transpose of the indicated vector. Letting $n = n_o + n_1$, the state space is $Y = L_2^2[0,2] \times E^n$ and, with $y(t) = (w(\cdot,t), v(\cdot,t), y_0(t), y_1(t))$ the system takes the form

$$\dot{y} = Fy + gu, \; y, \; g \in Y,$$

F being the operator defined by

$$F \begin{pmatrix} w \\ v \\ y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w' \\ v' \end{pmatrix} + A(\cdot) \begin{pmatrix} w \\ v \end{pmatrix} \\ A_0 + G_0 \begin{pmatrix} w(0) \\ v(0) \end{pmatrix} \\ A_1 y_1 + G_1 \begin{pmatrix} w(1) \\ v(1) \end{pmatrix} \end{pmatrix} \tag{2.17}$$

defined for $y_0 \;\; E^{n_0}$, $y_1 \in E^{n_1}$ and $(w,v) \in H_2^1[0,1]$ satisfying boundary conditions of the form (2.15), (2.16).

It would lead us too far astray to give a complete discussion of the eigenvalue problem for the operator F. For the case $A(x) \equiv 0$ however, the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w' \\ v' \end{pmatrix} = \lambda \begin{pmatrix} w \\ v \end{pmatrix}$$

can be solved explicitly to yield

$$w(x) = \alpha e^{\lambda x} + \beta e^{-\lambda x}, \; v(x) = \alpha e^{\lambda x} - \beta e^{-\lambda x} .$$

Combining these equations with

$$A_0 y_0 + G_0 \begin{pmatrix} w(0) \\ v(0) \end{pmatrix} = \lambda y_0, \quad A_1 y_1 + G_1 \begin{pmatrix} w(1) \\ v(0) \end{pmatrix} = \lambda y_1$$

we see that the eigenvalues of F are characterized as complex numbers for which the system of $n + 2$ equations in $n + 2$ unknowns

$$\begin{pmatrix} a_0 + b_0 & a_0 - b_0 & c_0^* & 0 \\ (a_1 + b_1)e^{\lambda} & (a_1 - b_1)e^{-\lambda} & 0 & c_1^* \\ G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} & G_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} & A_0 - \lambda I & 0 \\ G_1 \begin{pmatrix} e^{\lambda} \\ e^{\lambda} \end{pmatrix} & G_1 \begin{pmatrix} e^{-\lambda} \\ -e^{-\lambda} \end{pmatrix} & 0 & A_1 - \lambda I \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \hat{y}_0 \\ \hat{y}_1 \end{pmatrix} = 0$$

has a nontrivial solution $(\alpha, \beta, \hat{y}_o\ \hat{y}_1) \in E^{n+2}$. Standard algebraic operations lead to the characteristic equation

$$e^{\lambda}\left[\left((a_1 + b_1)p_1(\lambda) - q_1^0(\lambda)\right)\left((a_0 - b_o)\ p_o(\lambda) - q_2^1(\lambda)\right)\right]$$

$$-e^{-\lambda}\left[\left((a_0 + b_0)p_0(\lambda) - q_0^0(\lambda)\right)\left((a_o - b_1)p_1(\lambda) - q_1^1(\lambda)\right)\right] = 0\,, \qquad (2.18)$$

where $p_0(\lambda)$, $p_1(\lambda)$ are the characteristic polynomials of A_0, A_1 and $q_0^0(\lambda)$, $q_1^1(\lambda)$, $q_1^0(\lambda)$, $q_0^1(\lambda)$ are polynomials with degrees $\leq n_0 - 1$, $n_0 - 1$, $n_1 - 1$, $n_1 - 1$, respectively. If the polynomials $q_j^i(\lambda)$ were identically zero, the solutions of (2.18) would be those of (cf. (2.3))

$$e^{2\lambda} = \gamma$$

together with the zeros of $p_0(\lambda)$, $p_1(\lambda)$. For general $q_j^i(\lambda)$ one may use Rouché's theorem to see that the solution of (2.18) are asymptotic to those obtained in this simpler case. A slightly more sophisticated use of Rouché's theorem allows one, in fact, to see that this remains true even for a general continuous 2×2 matrix in (2.12).

To summarize, the eigenvalues of the operator F have

Property E The eigenvalues of F consist of complex numbers

$$\lambda_j = \frac{1}{2} \log \gamma + j\pi i + \varepsilon_j, \; -\infty < j < \infty \tag{2.19}$$

with γ as in (2.3), and

$$\sum_{j=-\infty}^{\infty} |\varepsilon_j|^2 < \infty,$$

together with $n = n_0 + n_1$ additional complex numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$.

At most finitely many of these eigenvalues may have multiplicity > 1. Without loss of generality they may be included in (2.19), repeated as often as their multiplicity indicates, since finite repetitions or the exchange of elements between the sets $\{\lambda_j\}$, $\{\sigma_k\}$ does not disturb the asymptotic relationship exhibited in (2.19). It should be observed, however, that the $\{\sigma_k\}$ cannot be included in the $\{\lambda_j\}$ while maintaining (2.19).

It is not difficult to demonstrate the uniform basis property of the eigenfunctions of F -- at least when $A(x) \equiv 0$ and the eigenvalues are distinct -- but this would carry us too far astray to include it here. Thus, provided the control distribution element $(g(\cdot), g_0, g_1)$ is such that the condition corresponding

to (1.5) is met, (2.12) - (2.16) provides an example of an augmented hyperbolic system.

We turn now to an example, really an infinite class of examples, of systems of deficient hyperbolic type. In [0] there appears a study of the control system

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad (x,y)\in\Omega, \ t \geq 0, \tag{2.20}$$

$$\frac{\partial w}{\partial \nu} = g(x,y,t), \ (x,y)\in\Gamma, \ t \geq 0, \tag{2.21}$$

where Ω is the disc of unit radius in R^2 and Γ is its boundary, the unit circle. The control function $g(x,y,t)$ is in $L^2_{loc}(\Gamma \times [0,\infty))$.

If we change to polar coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$ and employ the usual separation of variables technique

$$w(r,\theta,t) = w_0(r,t) + \sum_{k=1}^{\infty} (w_k(r,t) \cos k\theta + w_k(r,t) \sin k\theta)$$

$$g(1,\theta,t) \equiv g(\theta,t) = u_0(t) + \sum_{k=1}^{\infty} (\cos k\theta u_k(t) + \sin k\theta\, v_k(t)),$$

$$\int_0^T |u_0(t)|^2\, dt + \sum_{k=1}^{\infty} \int_0^T \left(|u_k(t)|^2 + |v_k(t)|^2\right) dt < \infty, \quad T > 0,$$

there results an infinite collection of partial differential equations in r,t:

$$\frac{\partial^2 w_k(r,t)}{\partial t^2} - \frac{\partial^2 w_k(r,t)}{\partial r^2} - \frac{1}{r}\frac{\partial w_k(r,t)}{\partial r} + \frac{k^2}{r^2} w_k\ (r,t) = 0 \tag{2.22}$$

$$w_k(0+,t) \text{ bounded}, \quad \frac{\partial w_k}{\partial r}(1,t) = u_k(t), \quad k = 0,1,2,\ldots \tag{2.23}$$

$$\frac{\partial^2 \tilde{w}_k(r,t)}{\partial t^2} - \frac{\partial^2 \tilde{w}_k(r,t)}{\partial r^2} - \frac{1}{r}\frac{\partial \tilde{w}_k(r,t)}{\partial r} + \frac{k^2}{r^2}\tilde{w}_k(r,t) = 0 , \tag{2.24}$$

$$\tilde{w}_k(0+,t) \text{ bounded}, \quad \frac{\partial \tilde{w}_k}{\partial r}(1,t) = v_k(t), \quad k = 1,2,\ldots \tag{2.25}$$

Let us study one of the systems (2.22), (2.23) for a fixed positive integer k. We may represent the homogeneous system in first order form by

$$\frac{\partial}{\partial t}\begin{pmatrix} y(\cdot,t) \\ z(\cdot,t) \end{pmatrix} = \begin{pmatrix} z(\cdot,t) \\ -Ly(\cdot,t) \end{pmatrix} \equiv A\begin{pmatrix} y(\cdot,t) \\ z(\cdot,t) \end{pmatrix} \tag{2.26}$$

where

$$y(\cdot,t) = w_k(\cdot,t), \quad z(\cdot,t) = \frac{\partial w_k}{\partial t}(\cdot,t), \tag{2.27}$$

$$(Ly)(r) = -\frac{\partial^2 y}{\partial r^2} - \frac{1}{r}\frac{\partial y}{\partial r} + \frac{k^2}{r^2}y \tag{2.28}$$

is an unbounded positive self adjoint operator on the Hilbert space

$$H_r = \{y \text{ measurable } | \int_0^1 r|y(r)|^2\, dr < \infty \}$$

with inner product

$$(y_1,y_2)_{H_r} = \int_0^1 r\, y_1(r)\, \overline{y_2(r)}\, dr.$$

The operator A generates a strongly continuous group, $S(t)$, of bounded operators on the Hilbert space

$$H = \{(y,z) \mid \int_0^1 r\left\{\frac{k^2}{r^2}|y(r)|^2 + |y'(r)|^2 + |z(r)|^2\right\} dr < \infty\}$$

with inner product

$$((y_1,z_1),(y_2,z_2))_H = \int_0^1 r\left\{\frac{k^2}{r^2} y_1(r)\,\overline{y_2(r)} + y_1'(r)\,\overline{y_2'(r)} + z_1(r)\,\overline{z_2(r)}\right\} dr$$

It is well known ([4], [7]) that the eigenvalues of the operator L are the numbers

$$\nu_{k\ell} = (\omega_{k\ell})^2, \quad \ell = 1,2,\ldots,$$

where $\omega_{k\ell}$ is the ℓ-th zero of the Bessel function $J_k(r)$ of first kind and order k. The corresponding normalized eigenfunctions

$$R_{k\ell}(r) = \left[\frac{2\nu_{k,\ell}}{(\nu_{k,\ell} - k^2)(J_k(\omega_{k\ell}))^2}\right]^{\frac{1}{2}} J_k(\omega_{k\ell} r)$$

satisfy

$$\left(R_{k_{\ell 1}}, R_{k_{\ell 2}}\right)_{H_r} = \delta_{\ell_1 \ell_2} .$$

The operator A has eigenvalues

$$\lambda_{k,\ell} = \begin{cases} i\omega_{k\ell}, & \ell = 1,2,\ldots \\ -i\omega_{k,-\ell}, & \ell = -1, -2,\ldots \end{cases}$$

and the corresponding eigenvectors of A:

$$\begin{pmatrix} \varphi_{k,\ell}(r) \\ \psi_{k,\ell}(r) \end{pmatrix} = \begin{pmatrix} \dfrac{-\operatorname{sgn}(\ell)\, R_{k\ell}(r)}{i\omega_k} \\ R_{k\ell}(r) \end{pmatrix}, \quad \ell = \pm 1, \pm 2, \ldots$$

may be seen to form a uniform basis for H. From the fact that $\big((y(t), z(t)),$ $(\hat{y}(t), \hat{z}(t))\big)_H$ is constant when (y,z), $(\hat{y},\hat{z})$ are solutions of (2.26), (2.27), it may be inferred that A is an unbounded antihermitian operator on H, i.e.,

$$A^* = -A.$$

The domain of A consists of (y,z) such that $(z,Ly) \in H$, $z(0+)$ bounded, $y(0+)$ bounded, $y'(1) = 0$.

Suppose $w_k(r,t)$, $\hat{w}_k(r,t)$ satisfy (2.22) and (2.23) and the corresponding homogeneous system with $u_k(t) \equiv 0$, respectively. Defining $y, z, \hat{y}, \hat{z}$ as in (2.27), for initial states $(y(0), z(0))$, $(\hat{y}(0), \hat{z}(0)$ in the domain of A and smooth $u_k(t)$ it may be verified that

$$\frac{d}{dt}\big((y(\cdot,t), z(\cdot,t)),\ (\hat{y}(\cdot,t),\ \hat{z}(\cdot,t))\big)_H = \hat{z}(1,t)u_k(t).$$

Thus the inhomogeneous system (2.22), (2.23) may be represented in the form

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = A\begin{pmatrix} y \\ z \end{pmatrix} + bu(t) \tag{2.29}$$

where b is the linear functional defined on the domain of $A^* (= -A)$ by

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, b \right\rangle = \hat{z}(1) .$$

The control distribution coefficients are the numbers

$$\left\langle \overline{\begin{pmatrix} \dfrac{-\mathrm{sgn}\ (\ell)\ R_{k,\ell}}{i\omega_{k,\ell}} \\ R_{k,\ell} \end{pmatrix}}, \ b \right\rangle = R_{k\ell}(1) \tag{2.30}$$

If $w_k(r,t)$ is represented as

$$w_k(r,t) = \sum_{\ell=1}^{\infty} y_\ell(t)\ R_{k\ell}(r)$$

$$\frac{\partial w_k}{\partial t}(r,t) = \sum_{\ell=1}^{\infty} z_\ell(t)\ R_{k\ell}(r)$$

it may be seen that

$$\left\|\begin{pmatrix} w_k(r,t) \\ \dfrac{\partial w_k}{\partial t}(r,t) \end{pmatrix}\right\|_H^2 = \sum_{\ell=1}^{\infty} \left(\nu_{k,\ell}\, |y_\ell(t)|^2 + |z_\ell(t)|^2 \right)$$

and (2.29) is equivalent to

$$\frac{d}{dt}\begin{pmatrix} y_\ell \\ z_\ell \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\nu_{k\ell} & 0 \end{pmatrix}\begin{pmatrix} y_\ell \\ z_\ell \end{pmatrix} + \begin{pmatrix} 0 \\ R_k\ (1) \end{pmatrix} u_k(t), \tag{2.31}$$

$$= 1,2,\ldots$$

Letting ω_k, $y_\ell = \tilde{\eta}_\ell$, $z_\ell = \tilde{\zeta}_\ell$,

(2.31) becomes

$$\frac{d}{dt}\begin{pmatrix}\tilde{\eta}_\ell \\ \tilde{\zeta}_\ell\end{pmatrix} = \begin{bmatrix} 0 & \omega_{k,\ell} \\ -\omega_{k,\ell} & 0 \end{bmatrix}\begin{pmatrix}\hat{\eta}_\ell \\ \tilde{\zeta}_\ell\end{pmatrix} + \begin{pmatrix} 0 \\ R_{k,\ell(1)}\end{pmatrix} u_k(t)$$

and then with

$$\begin{pmatrix}\tilde{\eta}_\ell \\ \tilde{\zeta}_\ell\end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}\begin{pmatrix}\eta_\ell \\ \zeta_\ell\end{pmatrix}$$

we finally have the form equivalent to (1.10) of Section 1

$$\frac{d}{dt}\begin{pmatrix}\eta_\ell \\ \zeta_\ell\end{pmatrix} \quad \begin{bmatrix} i\omega_{k\ell} & \\ 0 & -i\omega_{k\ell} \end{bmatrix}\begin{pmatrix}\eta_\ell \\ \zeta_\ell\end{pmatrix} + R_{k\ell}(1)\begin{bmatrix} \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} u_k(t) \tag{2.32}$$

For a given $k = 1,2,\ldots$ it may be verified (see [4], [27]) that the $R_{k\ell}(1)$ are uniformly bounded and uniformly bounded away from zero. As for the $\omega_{k,\ell}$, it is known ([7],[27]) that they are separated by an interval of at least π. It follows that the hypotheses of Theorem 1.3 are all satisfied and b, appearing in (2.29), is an admissible control distribution element, the solution of (2.32) being continuous in ℓ^2 for an initial state in ℓ^2 and u_k locally square integrable. Since

$$\left\| \begin{matrix} w_k(\cdot,t) \\ \frac{\partial w_k}{\partial t}(\cdot,t) \end{matrix} \right\|_H^2 = \sum_{\ell=1}^{\infty} \left(|\eta_\ell|^2 + |\zeta_\ell|^2\right),$$

we conclude that the solution of (2.29) is continuous in H for any initial state in H and u locally square integrable. Finally we construct from the systems (2.22),(2.23) certain composite systems which are deficient hyperbolic

systems. We have already noted that the zeros of $J_k(r)$ are separated by an interval of length at least equal to π. In fact, the interval approaches π in length since ([27], Chap. XV, Section 15.53) the ℓ-th positive zero of $J_k(r)$ is given asymptotically by

$$\omega_{k,\ell} = (\ell + \frac{1}{2} k - \frac{1}{4})\pi + \mathcal{O}(\frac{1}{\ell}), \tag{2.33}$$

and there are (Ibid., Section 15.4) exactly ℓ positive zeros of $J_k(r)$ between the imaginary axis and the line $\mathrm{Re}(r) = (\ell + \frac{1}{2} k + \frac{1}{4})\Pi$. The presence of the term $\frac{1}{2} k$ here shows that, to include a fixed number of zeros, progressively longer intervals must be used as k, the order of the Bessel function involved, gets larger. This indicates that we might expect the systems to be deficient with ever higher indices of deficiency as k increases. Unfortunately, no single system (2.22),(2.23) will serve as an example because, however deficient, exponentials $e^{\pm i\omega_{k,\ell} t}$, with the $\omega_{k,\ell}$ having the asymptotic form (2.33), cannot be included in a uniform basis by adjoining other exponentials. The case

$$\omega_{k,\ell} = (\ell + \frac{1}{2} k - \frac{1}{4})\pi$$

with k even is given as an example by Levinson in [12] to show that his famous $\pi/4$ limit on perturbations of exponents from integer multiples of π cannot be improved upon if the resulting exponentials are to form a uniform basis.

To construct our examples we group the systems (2.22), (2.23) in pairs $k = 0,1$, $k = 2,3,\ldots$, $k = 2j, 2j + 1,\ldots$. In the composite system formed from (2.22), (2.23) with $k = 2j, 2j + 1$ we require

$$u_{2j}(t) \equiv u_{2j+1}(t) \equiv U_j(t) .$$

We now have, in place of (2.32)

$$\frac{d}{dt}\begin{pmatrix}\eta_\ell\\ \zeta_\ell\\ \hat{\eta}_\ell\\ \hat{\zeta}_\ell\end{pmatrix}=\begin{pmatrix} i\omega_{2j,\ell} & 0 & 0 & 0\\ 0 & -i\omega_{2j,\ell} & 0 & 0\\ 0 & 0 & i\omega_{2j+1,\ell} & 0\\ 0 & 0 & 0 & -i\omega_{2j+1,\ell}\end{pmatrix}\begin{pmatrix}\eta_\ell\\ \zeta_\ell\\ \hat{\eta}_\ell\\ \hat{\zeta}_\ell\end{pmatrix}$$

$$+\begin{pmatrix} -\,i\,R_{2j,\ell}(1)/\sqrt{2}\\ i\,R_{2j,\ell}(1)/\sqrt{2}\\ -\,i\,R_{2j+1,\ell}(1)/\sqrt{2}\\ i\,R_{2j+1,\ell}(1)/\sqrt{2}\end{pmatrix} U_j(t),\quad \ell = 1,2,3\ldots \tag{2.34}$$

For each j the totality of exponents appearing on the *diagonals* of the matrices in (2.34) for $= 1,2,3,\ldots$ constitute a set $\pm\, i\hat{\omega}_{j,n}$ with

$$\hat{\omega}_{j,n} = (2j + n + \tfrac{1}{2})\,\frac{\pi}{2} + \mathcal{O}(\tfrac{1}{n}),\quad n = 1,2,3,\ldots$$

Here the $\pm\,\hat{\omega}_{j,n}$ all differ, asymptotically, by multiples of $\frac{\pi}{2}$ and the set of exponentials $e^{\pm i\hat{\omega}_{j,n} t}$ can be augmented to form a uniform basis for $L^2[-2,2]$. The number of exponentials which must be adjoined is $4j+2$, $j = 0,1,2,\ldots$. Thus the j-th system (2.34) is a deficient hyperbolic system with index of deficiency $4j + 2$.

3. The Spaces $H^m[a,b]$ and their Dual Spaces

We introduce here some spaces which will be central to the work of this paper and establish some of their properties. By $H^m[a,b]$, m an integer ≥ 0, we designate the space of complex valued functions $z:[a,b] \to \mathbb{C}$ which, together with their derivatives $z^{(k)}$ $k = 1,2,\ldots, m$, defined in the sense permitted by the theory of distributions, lie in $L^2[a,b] \equiv H^0[a,b]$. It is well known [1] that these are Hilbert spaces with the inner product and norm

$$[z,\hat{z}]_m = \sum_{k=0}^{m} \int_a^b z^{(k)}(t)\, \overline{\hat{z}^{(k)}}(t)\, dt, \tag{3.1}$$

$$|z|_m = ([z,z]_m)^{\frac{1}{2}} \tag{3.2}$$

This is the traditional formulation of these spaces but it is not particularly convenient in applications. The inconvenience lies in the fact that the "coordinate" functions $z^{(k)}$, $k = 0,1,2,\ldots,m$, cannot be independently specified. This leads to substantial computational difficulty in computing adjoints of operators, for example. Thus, rather than representing elements of $H^m[a,b]$ in terms of the n-tuple of functions $(z,z',\ldots,z^{(m)})$, we select a point $c \epsilon [a,b]$ and represent z by

$$(z(c),\ z'(c),\ldots,\ z^{(m-1)}(c),\ z^m(\cdot)),$$

the first m of these being complex numbers, the last a function in $L^2[a,b]$. It is clear that each element z of $H^m[a,b]$ may be uniquely specified this way and that $z(c)$, $z'(c),\ldots,$ $z^{(m-1)}(c)$, $z^m(\cdot))$ may be selected independently to construct elements $z \in H^m[a,b]$. This shows $H^m[a,b]$ to be equivalent to the

space $E^m \times L^2[a,b]\cdot$, indeed, the map $F: E^m \times L^2[a,b] \to H^m[a,b]$:

$$F(h_0, h_1, \ldots, h_{m-1}, h_m(\cdot))(t) =$$

$$h_0 + h_1(t-c) + h_2(t-c)^2/2 + \ldots + h_{m-1}(t-c)^{m-1}/(m-1)!$$

$$+ \frac{(-1)^{m-1}}{(m-1)!} \int_0^t (s-t)^{m-1} h_m(s)\, ds, \quad t \in [a,b],$$

defines an algebraic and topological isomorphism between the two spaces. It follows that the inner product in $H^m[a,b]$ defined by

$$(z,\hat{z})_m = \sum_{k=0}^{m-1} z^{(k)}(c)\, \overline{\hat{z}^{(k)}}(c) + \int_a^b \hat{z}^{(m)}(t)\, \overline{\hat{z}^{(m)}}(t) \tag{3.3}$$

is such that the related norm

$$\| z \|_m = ((z,z)_m)^{\frac{1}{2}} \tag{3.4}$$

defines a topology $H^m[a,b]$ equivalent to the one defined by (3.1),(3.2).

The dual space $H^m[a,b]^*$, the space of continuous linear functionals on $H^m[a,b]$, may, of course, be identified with $H^m[a,b]$ itself. We choose not to do this, however; instead we follow a procedure now familiar from the work [13] of Lions-Wagenes and others[11],[15]. The map $J: H^m[a,b] \to L^2[a,b]$ defined as the injection of $H^m[a,b]$ into the larger space $L^2[a,b]$ is clearly continuous and one-to-one with dense range in $L^2[a,b]$. It follows that for each $f \in L^2[a,b]$

$$\langle z, f \rangle \equiv \int_a^b z(t)\, \overline{f(t)}\, dt = (z,f)_{L^2[a,b]}, \quad z \in H^m[a,b],$$

defines a continuous linear functional on $H^m[a,b]$. We define

$$\|f\|_{-m} = \sup_{\substack{z \in H^m[a,b] \\ z \neq 0}} \left\{ \frac{|\langle z,f \rangle|}{\|z\|_m} \right\} ,$$

and verify easily that $\| \ \|_{-m}$ defines a norm on $L^2[a,b]$. Except for the case $m = 0$, $L^2[a,b]$ is not complete with respect to this norm. We define $D^m[a,b]$, the dual of $H^m[a,b]$ with respect to $L^2[a,b]$, to be the completion of $L^2[a,b]$ with respect to this norm. Each element of $D^m[a,b]$ thus corresponds to a sequence $\{f_k\} \subseteq L^2[a,b]$, Cauchy with respect to $\| \ \|_{-m}$. The norm in $D^m[a,b]$ is the one induced by $\| \ \|_{-m}$ as already defined on $L^2[a,b]$:

$$\|\phi\|_{-m} \quad \lim_{k \to \infty} \|f_k\|_{-m} , \tag{3.5}$$

here ϕ is the element of $D^m[a,b]$ identified with the sequence $\{f_k\}$ just described. The bilinear form $\langle z,\phi \rangle$ is defined for $\phi \in D^m[a,b]$ by

$$\langle z,\phi \rangle \equiv \lim \langle z,f_k \rangle . \tag{3.6}$$

It may be verified that (3.5), (3.6) are independent of the choice of $\{f_k\}$ within the class of equivalent Cauchy sequences. It is known that $D^m[a,b]$, as just constructed, is algebraically and topologically and isomorphic to $H^m[a,b]^*$. One may identify $D^m[a,b]$ with a certain vector space of distributions: those having the form (not unique; c could be replaced by any other point in $[a,b]$, e.g.)

$$\phi = d_v^{(m)} + c_o \delta_c + c_1 \delta'_c + \ldots + c_{m-1} \ c^{(m-1)} ,$$

Here $v \in L^2[a,b]$ and for each $z \in H^m[a,b]$

$$(z, d_v^{(m)}) = \int_a^b z^{(m)}(s)\, \bar{v}(s)\, ds,$$

while δ_c, δ_c', etc., refer to the Dirac distribution with support at a point $c \in [a,b]$ and its distributional derivatives.

We proceed next to define the spaces $H^m[a,b]$ for $m > 0$. To avoid confusion it is preferable to take $m > 0$ and refer to the space as $H^{-m}[a,b]$. The definition begins with specification of $D^{-m}[a,b]$:

$$D^{-m}[a,b] = \{z \in H^m[a,b] \mid z(a) = z(b) = z'(a) = z'(b)$$

$$= \ldots = z^{(m-1)}(a) = z^{(m-1)}(b) = 0\} \quad .$$

This space, of course, is often referred to as $H_0^m[a,b]$. Again the injection $\hat{J}: D^{-m}[a,b] (= H_0^m[a,b]) \to L^2[a,b]$ is continuous and one to one with dense range. We define $H^{-m}[a,b]$, $m = 1,2,3,\ldots,$ to be the dual of $D^{-m}[a,b]$ with respect to $L^2[a,b]$, formed in the same way as $D^m[a,b]$ was earlier constructed as the dual of $H^m[a,b]$ with respect to $L^2[a,b]$. It is not hard to see that $H^{-m}[a,b]$ is a subspace of $D^m[a,b]$ complementary to the 2m-dimensional subspace of $D^m[a,b]$ spanned by the Dirac distributions $\delta_a, \delta_b, \delta_a', \delta_b', \ldots, \delta_a^{(m-1)}, \delta_b^{(m-1)}$.

We are particularly concerned with the construction of uniform bases for the spaces $H^m[a,b]$, $H^{-m}[a,b]$ which consist of generalized exponentials $e^{\lambda t}, te^{\lambda t}, \frac{t^2}{2} e^{\lambda t}, \ldots$. To carry out this study it will be convenient to normalize the interval involved by supposing that $a = -1$, $b = 1$, and $c = 0$. The reader will be familiar with the standard results ([14],[12],[24],[19],[17]) concerning "nonharmonic Fourier series" in $L^2[-1,1]$. Suppose $\{\lambda_j \mid -\infty < j < \infty\}$ is a sequence of distinct complex numbers such that

$$\inf_{\hat{J}} \left\{ \sup_{j \in \hat{J}} |\lambda_j - (\alpha + j\pi i)| \right\} \le \rho < \frac{\pi}{4}, \tag{3.7}$$

where $\hat{J}$ denotes any finite subset of $J = \{j \mid -\infty < j < \infty\}$ and α is an arbitrary, but fixed relative to j, complex number. Then the exponentials $e^{\lambda_j t}$ form a uniform basis (cf. (1.2), (1.3)) for $L^2[-1,1]$. It is known that the $\pi/4$ in (3.7) is best possible [12]. It is clear that (3.7) is true if the λ_j have the property

$$\lambda_j = \alpha + j\pi i + \varepsilon_j, \quad \sum_{j=-\infty}^{\infty} |\varepsilon_j|^2 < \infty,$$

and this is the case of greatest interest to us. There are many instances of uniform bases of exponentials where the λ_j do not satisfy (3.7). We hope to discuss some of these in another article. It is not hard to show using the Fourier transform methods of [12], [14] that finitely many exponentials $e^{\lambda_j t}$ may be replaced by an equal number of generalized exponentials

$$\begin{array}{l} e^{\lambda_{j_1} t}, \; te^{\lambda_{j_1} t}, \ldots, \dfrac{t^{\mu_1 - 1}}{(\mu_1 - 1)!} e^{\lambda_{j_1} t} \\ \vdots \\ e^{\lambda_{j_r} t}, \; te^{\lambda_{j_r} t}, \ldots, \dfrac{t^{\mu_r - 1}}{(\mu_r - 1)!} e^{\lambda_{j_r} t}, \end{array}$$

provided that all λ_j remain distinct, without changing the uniform basis property. There are cases (see e.g. [26] of uniform bases containing infinitely many generalized exponentials. In all cases it is assumed that when $\frac{t^n}{n!} e^{\lambda t}$ is present, then $\frac{t^j}{j!} e^{\lambda t}$ are also present for $j = 0,\ldots,n-1$.

Our immediate objective is to extend these results to the spaces $H^m[-1,1]$

and $H^{-m}[-1,1]$, $m = 1,2,3,\ldots$. We will state and prove the theorem for the case of distinct exponential functions but the results remain true where generalized exponentials are present also. We are reserving detailed discussion of generalized exponentials to Section 7.

<u>Theorem 3.1</u> <u>Let</u> J <u>be a general countable index set, as used in Sections 1 and 2, and let</u> $\Lambda = \{\lambda_j | j \in J\}$ <u>be a sequence of distinct complex numbers such that</u> (i) $E_\Lambda \equiv \{e^{\lambda_j s} | j \in J\}$ <u>forms a uniform basis for</u> $L^2[-1,1]$ <u>and</u> (ii) $\sum_{j\in J} (|\lambda_j| + 1)^{-2} < \infty$. <u>Let</u> $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m$ <u>be a set of distinct complex numbers not included in</u> Λ. <u>Then</u>

$$E_{\Lambda,\Sigma} = \{e^{\sigma_k s},\ e^{\lambda_j s}/p(\lambda_j) | j \in J,\ \ k = 1,2,\ldots,m\}, \tag{3.8}$$

$$p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k), \tag{3.9}$$

<u>is a uniform basis for</u> $H^m[-1,1]$.

<u>Proof</u> In every case known to this author where (i) is true, (ii) is also true. Since a cursory search of the literature reveals no theorem precisely to this effect, however, we state it as an assumption here.

<u>Proof of Theorem 3.1</u> We form the m-th order differential operator L defined by

$$L\omega = \prod_{k=1}^{m} (D - \sigma_k I)\omega , \tag{3.10}$$

where $(D\omega)(s) = \dfrac{d\omega(s)}{ds}$, $s \in [-1, 1]$. Then for an arbitrary $z \in H^m[-1,1]$,

$$Lz = f \in L^2[-1,1]$$

and, from the uniform basis property of E_Λ

$$f(s) = \sum_{j\in J} f_j e^{\lambda_j s}, \quad \sum_{j\in J} |f_j|^2 < \infty , \tag{3.11}$$

the series convergent in $L^2[-1,1]$ and (cf. (1.3))

$$d^{-2} \|f\|^2_{L^2[-1,1]} \le \sum_{j\in J} |f_j|^2 \le D^2 \|f\|^2_{L^2[-1,1]} . \tag{3.12}$$

Now the inhomogeneous differential equation $Lz = f$ has the particular solution

$$\hat{z}(s) = \sum_{j\in J} \frac{f_j}{p(\lambda_j)} e^{\lambda_j s} , \tag{3.13}$$

where $p(\lambda)$, given by (3.9), is the characteristic polynomial of L shown in (3.10). Then

$$z = \hat{z} + \tilde{z}$$

where $\tilde{z}$ is the solution of $L\tilde{z} = 0$ with

$$\tilde{z}(0) = z(0) - \hat{z}(0) . \tag{3.14}$$

Thus

$$\tilde{z}(s) = \sum_{k=1}^{m} c_k e^{\sigma_k s} \tag{3.15}$$

where the c_k are chosen so that (3.15) satisfies (3.13), i.e., using the familiar Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1 & \sigma_2 & \cdots & \sigma_m \\ \vdots & \vdots & & \vdots \\ (\sigma_1)^{m-1} & (\sigma_2)^{m-1} & \cdots & (\sigma_m)^{m-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} z(0) - \hat{z}(0) \\ z'(0) - \hat{z}'(0) \\ \vdots \\ z^{(m-1)}(0) - \hat{z}^{(m-1)}(0) \end{pmatrix} \tag{3.16}$$

Combining (3.13) and (3.15) we have

$$z(s) = \sum_{k=1}^{m} c_k e^{\sigma_k s} + \sum_{j \in J} \frac{f_j}{p(\lambda_j)} e^{\lambda_j s} . \tag{3.17}$$

Using the convergence of (3.11) in $L^2[-1,1]$ together with familiar regularity theorems for solutions of $Lz = f$(details below) we are able to conclude that the series (3.17) converges in $H^m[-1,1]$. Thus the set $E_{\Lambda,\Sigma}$ is complete in that space.

It remains to establish the property (1.3). From the uniform basis property of E_Λ in $L^2[-1,1]$ we have (3.12) for positive numbers d,D. Let

$$z_K(s) = \sum_{k=1}^{m} c_k e^{\sigma_k s} + \sum_{|j| \leq K} \frac{f_j}{p(\lambda_j)} e^{\lambda_j s} \tag{3.18}$$

be the solution of

$$Lz_K = f_K \equiv \sum_{|j| \leq K} f_j e^{\lambda_j s}$$

such that, with

$$\hat{z}_K(s) = \sum_{|j|\le K} f_j \frac{e^{\lambda_j s}}{p(\lambda_j)}$$

we have

$$z_K^{(\ell)}(0) - \hat{z}_K^{(\ell)}(0) = z^{(\ell)}(0) - \hat{z}^{(\ell)}(0), \; \ell = 0,1,\ldots, m-1. \quad (3.19)$$

(3.19) is equivalent to the c_k in (3.18) being independent of K and the same as those appearing in (3.17). Then for $\ell = 0,1,2\ldots, m-1$,

$$z_K^{(\ell)}(0) = \sum_{k=1}^{m} (\sigma_k)^{\ell} c_k + \sum_{|j|\le K} (\lambda_j)^k \frac{f_j}{p(\lambda_j)} .$$

From our assumption (ii) the sequences $\{\lambda_j^{(\ell)}/p(\lambda_j) | j \in J$ are in ℓ^2 for $\ell = 0,1,2,\ldots, m-1$. Using the Schwartz inequality in ℓ^2 we have

$$|z_K^{(\ell)}(0)|^2 \le \tilde{C}^{(\ell)} \left(\sum_{k=1}^{m} |c_k|^2 + \sum_{j=1}^{K} |f_j|^2 \right), \ell = 0,1,\ldots, m-1, \quad (3.20)$$

for each finite K, where $\tilde{C}^{(\ell)}$ is a positive constant independent of K. Now the regularity theorems for ordinary differential equations imply that

$$c_0^{-2} \|z_K\|^2_{H^m[-1,1]} \le \sum_{k=0}^{m-1} |z_K^{(\ell)}(0)|^2 + \|f_K\|^2_{L^2[-1,1]}$$

$$\le C_0^2 \|z_K\|^2_{H^m[-1,1]} \quad (3.21)$$

where c_0, C_0 depend only on the differential operator L, i.e., on $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Letting $K \to \infty$, using (3.20), the first inequality in (3.21) and comparable inequalities with z_K replaced by $z_K - z_{\hat{K}}$ K, $\hat{K}$ arbitrary positive integers, we have the result used earlier, to the effect that (3.17) in convergent in $H^m[-1,1]$ and

$$c^{-2} \|z\|^2_{H^m[-1,1]} \leq \sum_{k=1}^{m} |c_k|^2 + \sum_{j \in J} |f_j|^2 \tag{3.22}$$

for some $c > 0$. *To obtain the corresponding* inequality in the opposite direction, we note first of all that

$$\|f\|^2_{L^2[-1,1]} = \|Lz\|^2_{L^2[-1,1]}$$

so that, using (3.12)

$$\sum_{j \in J} |f_j|^2 \leq D^2 \|Lz\|^2_{H^m[-1,1]} .$$

But, from the form of the operator L it is clear that for some $B > 0$

$$\|Lz\|^2_{L^2[-1,1]} \leq B^2 \|z\|^2_{H^m[-1,1]} .$$

Thus

$$\sum_{j \in J} |f_j|^2 \leq D^2 B^2 \|z\|^2_{H^m[-1,1]} . \tag{3.23}$$

From (3.16) and the nonsingularity of the Vandermonde matrix

$$\sum_{k=1}^{m} |c_k|^2 \le B^2 \left[\sum_{\ell=0}^{m-1} |z^{(\ell)}(0)|^2 + \sum_{\ell=0}^{m=1} |\hat{z}^{(\ell)}(0)|^2 \right]$$

$$\le \tilde{B}^2 \|z\|^2_{H^m[-1,1]} + \hat{B}^2 \sum_{\ell=0}^{m-1} |\hat{z}^{(\ell)}(0)|^2 \qquad (3.24)$$

for some $\tilde{B}$, $\hat{B} > 0$. But, using the square summability of $(\lambda_j)^\ell / p(\lambda_j)$, $\ell = 0,1,2,\ldots, m-1$, as we did in (3.19), ff., together with the formula (3.13) for $z(s)$, we have

$$|\hat{z}^{(\ell)}(0)|^2 \le \tilde{c}^{(\ell)} \sum_{j\in J} |f_j|^2 . \qquad (3.25)$$

Using (3.24), (3.25) together

$$\sum_{k=1}^{m} |c_k|^2 \le \tilde{B}^2 \|z\|^2_{H^m[-1,1]} + \hat{B}^2 \sum_{\ell=0}^{m-1} c^{(\ell)} \sum_{j\in J} |f_j|^2 . \qquad (3.26)$$

Finally, using (3.23) twice, (3.26) gives

$$\sum_{k=1}^{m} |c_k|^2 + \sum_{j\in J} |f_j|^2 \le C^2 \|z\|^2_{H^m[-1,1]} . \qquad (3.27)$$

Since each $z \in H^m[-1,1]$ has the unique representation (3.17) with (3.22) and (3.27) valid, we conclude that the functions $e^{\sigma_k t}$, $k = 1,2,\ldots, m$, $e^{\lambda_j t}/p(\lambda_j)$, $j\in J$, form a uniform basis for $H^m[-1,1]$ and the proof is complete.

We turn next to a comparable result for the spaces $H^{-m}[-1,1]$, $m = 1,2,3,\ldots$. Our theorem here bears the same relationship to deficient hyperbolic systems as the preceding theorem does to augmented hyperbolic systems, both relationships to be exploited further in later sections.

Theorem 3.2 Let $\{\lambda_j | j \epsilon J\}$ be a sequence of complex numbers with the properties described in Theorem 3.1. Let J_m be a subset of J obtained by removing (any) m of these numbers; let that set be designated as $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. Then the set of functions

$$E_{\Lambda-\Sigma} \equiv \{p(\lambda_j) e^{\lambda_j s} | j \epsilon Jm\}$$

is a uniform basis for $H^{-m}[-1,1]$.

Proof We begin with consideration of $D^{-m}[-1,1] = H_0^m[-1,1]$ as described earlier. We let

$$Ly = \prod_{k=1}^{m} (D - \sigma_k I) y, \quad \overline{L}y = \prod_{k=1}^{m} (D - \overline{\sigma}_k I).$$

Now a function $y \epsilon D^{-m}[-1,1] = H_0^m[-1,1]$ is completely determined by $\overline{L}y$. Indeed the map $F: f \epsilon L^2[-1,1] \rightarrow y\ H^m[-1,1]$ defined by

$$(\overline{L}y)(s) = f(s),\ s \epsilon [-1,1],\ y^{(k)}(-1) = 0,\ k = 0,1,2,\ldots, m-1 \quad (3.28)$$

is bounded and one to one and boundedly invertible on its range. The further restriction necessary to obtain $y \epsilon H_0^m[-1,1]$

$$y^{(k)} = 0,\ k = 0,1,2,\ldots, m-1 \quad ,$$

imposes certain restrictions on f. We are still taking the λ_j to be distinct so (3.28) is equivalent, with $\eta_k = y^{(k-1)}$, $k = 1,2,\ldots, m$, to the m-dimensional *first order equation*

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_m\end{pmatrix} = \begin{pmatrix}\bar{\sigma}_1 & 0 & \cdots & \eta_1\\ 0 & \bar{\sigma}_2 & \cdots & \eta_2\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\sigma}_m\end{pmatrix} + \begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{pmatrix} f ,$$

with

$$\eta_k(-1) = 0, \; k = 1,2,\ldots, m. \tag{3.29}$$

It may be seen that $b_k \neq 0$, $k = 1,2,\ldots, m$. From the variation of parameters formula and (3.29)

$$\eta^{(k-1)}(1) = b_k \int_{-1}^{1} e^{\bar{\sigma}_k(1-s)} f(s)ds, \; k = 1,2,\ldots, m$$

and we conclude $y \in H_0^m[-1,1]$ just in case

$$\int_{-1}^{1} e^{-\bar{\sigma}_k s} f(s)ds = 0, \; k = 1,2,\ldots, m. \tag{3.30}$$

Since the derivatives of order $<m$ of $y \in H_0^m[-1,1]$ can be expressed in terms of continuous linear operations on $\bar{L}y$, every continuous linear functional z on $H_0^m[-1,1]$ can be expressed in the form

$$\langle y,z\rangle = \int_{-1}^{1} \overline{\zeta(-s)}\, \bar{L}y(s)ds, \tag{3.31}$$

for some $\zeta \in L^2[-1,1]$. Then

$$\zeta = \sum_{j\in J} \zeta_j e^{\lambda_j s} \tag{3.32}$$

with

$$d^{-2} \|\zeta\|^2_{L^2[-1,1]} \leq \sum_{j\epsilon J} |\zeta_j|^2 \leq D^2 \|\zeta\|^2_{L^2[-1,1]} \tag{3.33}$$

Since $\bar{L}y = f$ satisfies (3.30), however, the terms involving $e^{\sigma_k s}$ in (3.32) make *no* contribution in (3.31) and we have, equivalently,

$$\langle y,z \rangle = \int_{-1}^{1} \overline{\tilde{\zeta}(-s)}\, \bar{L}y(s)ds,$$

$$\tilde{\zeta} = \sum_{j\epsilon J_m} \zeta_j e^{\lambda_j s} \tag{3.34}$$

and

$$d^{-2} \|\tilde{\zeta}\|^2_{L^2[-1,1]} \leq \sum_{j\epsilon J_m} |\zeta_j|^2 \leq D^2 \|\tilde{\zeta}\|^2_{L^2[-1,1]} \quad . \tag{3.35}$$

Now let ω be an arbitrary element of $L^2[-1,1]$. Consider the linear functional defined for $y \epsilon L^2[-1,1]$ by

$$\int_{-1}^{1} \overline{\omega(-s)} y(s)\, ds \quad .$$

The function $\overline{\omega(-s)}$ can be expressed as

$$\omega(-s) = (L\zeta)(-s),\ \zeta\epsilon H^m[-1,1] \subseteq L^2[-1,1], \tag{3.36}$$

$$\zeta(s) = \tilde{\zeta}(s) + \sum_{k=1}^{m} c_k e^{\sigma_k s},\ \tilde{\zeta}\epsilon H^m[-1,1], \tag{3.37}$$

where the c_k are indeterminate. Now suppose we consider only $y \epsilon H_0^m[-1,1]$. Noting that

$$\int_{-1}^{1} \overline{\zeta^{(k)}(-s)}\, y(s)ds = \int_{-1}^{1} \overline{\zeta(-s)}\ \frac{d^k y(s)}{ds^k}\, ds,\ k = 0,1,\dots, m,$$

for such functions, we have

$$\int_{-1}^{1} \overline{\omega(-s)}\, y(s)ds = \int_{-1}^{1} \overline{(L\zeta)(-s)}\, y(s)ds = \int_{-1}^{1} \overline{\zeta(-s)}\ Ly(s)ds$$

$$= \text{(from (3.30),(3.37))} = \int_{-1}^{1} \overline{\tilde{\zeta}(-s)}\ \bar{L}y(s)ds,$$

In $H_0^m[-1,1]$, $\|\bar{L}y\|_{L^2[-1,1]}$ is equivalent to $\|y\|_{H_0^m[-1,1]}$. It follows that the norm of $\tilde{\zeta}$ as a linear functional on $H_0^m[-1,1]$ defined by $\int_{-1}^{1} \tilde{\zeta}(-t)\bar{L}y(t)dt$ is equivalent to the $L^2[-1,1]$ norm of $\tilde{\zeta}$, which satisfies (3.35). We have seen, moreover, that we may as well assume that $\tilde{\zeta}$ is given by (3.34).

We define a norm $\|\!|\ \|\!|_{-m}$, equivalent to $\|\ \|_{-m}$, by

$$\|\!|\omega\|\!|_{H^{-m}[-1,1]} = \|\tilde{\zeta}\|_{L^2[-1,1]}\ . \tag{3.38}$$

In this topology it is clear from (3.34),(3.36),(3.37) that when the series (3.32) is a finite series, ω is not distinguished from

$$\hat{\omega}(s) = \sum_{j\in J_m} \zeta_j p(\lambda_j)e^{\lambda_j s},$$

which leads, by the above process, to the same $\tilde{\zeta}$. Using a continuity argument, this remains true for convergent series (3.32) which do not terminate. Thus we can write

$$\omega(s) = \sum_{j\in J_m} \zeta_j p(\lambda_j) e^{\lambda_j s} \tag{3.39}$$

as far as the topology (3.38) is concerned, and, from (3.35), (3.38)

$$d^{-2} \|\omega\|_{H^{-m}[-1,1]} = d^{-2} \|\tilde{\zeta}\|_{L^2[-1,1]} \leq \sum_{j\in J_m} |\zeta_j|^2 \leq D^2 \|\tilde{\zeta}\|_{L^2[-1,1]}$$

$$= D^2 \|\omega\|_{H^m[-1,1]} . \tag{3.40}$$

Since $H^m[-1,1]$ is the completion of $L^2[-1,1]$ with respect to the topology (3.38), it is clear that each $\omega \in H^{-m}[-1,1]$ corresponds to a series (3.39) with $\sum_{j\in J_m} |\zeta_j|^2 < \infty$ and each such series yields an element of that space. The inequalities (3.40) hold for all $\omega \in H^{-m}[-1,1]$ and the proof is complete.

4. Scalar Linear Neutral Equations of Finite Order

For n a non-negative integer, a scalar neutral functional equation of order n takes the form, for $a > 0$,

$$\sum_{k=0}^{m} c_k \zeta^{(k)} (t + a) + \sum_{k=0}^{m} d_k \zeta^{(k)} (t - a)$$

$$+ \sum_{k=0}^{m} \int_{-a}^{a} \zeta^{(k)} (t+ s) \, d\, v_k(s) = u(t), \tag{4.1}$$

where u is regarded as an **external** "forcing" or "control" term. The c_k, d_k are constants with

$$c_m \neq 0, \; d_m \neq 0 \tag{4.2}$$

and the v_k are functions of bounded variation on $[-a,a]$ which may be assumed right continuous. Associated with a function v of bounded variation on $[-a,b]$ and an interval $[\alpha,\beta] \subseteq [-a,a]$ we have the total variation (cf. [8],[18]), $V(v[\alpha,\beta])$, of v on $[\alpha,\beta]$. The basic requirement in order that (4.1) lead to well posed problems is

$$\lim_{\varepsilon \to 0} V(v_m[-a,-a +\varepsilon]) = \lim_{\varepsilon \to 0} V(v_m,[a -\varepsilon, a]) = 0 \tag{4.3}$$

We have seen in the preceding section that there are various ways in which the elements of the space $H^m[-a,a]$ may be specified. *There is comparable* ambiguity in the expression of a given neutral functional equation. This leads us to adopt a standard form compatible with the way in which we chose to express

elements of $H^m[a,b]$ in Section 3. First of all, each of the integrals

$$\int_{-a}^{a} \zeta^{(k)}(t+s)\, d\, v_k(s),\ k = 0,1,\ldots,m-1,$$

can be integrated by parts $m-k$ times to yield an integral of the form

$$\int_a^b \zeta^{(m)}(t+s)\, \tilde{v}_k(s)\, ds,\ v_k \in H^{m-k-1}[a,b],$$

plus boundary terms involving $\zeta^{(j)}(t-a)$, $\zeta^{(j)}(t+a)$, $j = k, k+1,\ldots, m-1$. As a result (4.1) can be rewritten as

$$C_m\, \zeta^{(m)}(t+a) + d_m\, \zeta^{(m)}(t-a) + \sum_{k=0}^{m-1} \left| \tilde{c}_k \zeta^{(k)}(t+a) + \tilde{d}_k\, \zeta^{(k)}(t-a) \right|$$

$$+ \int_{-a}^{a} \zeta^{(m)}(t+s)\, d\, v_m(s) + \int_{-a}^{a} \zeta^{(m)}(t+s)\, \tilde{v}(s) ds = u(t)$$

$$\tilde{v} = \sum_{k=0}^{m-1} \tilde{v}_k \in L^2[-a,a] \quad . \tag{4.4}$$

Next, selecting an arbitrary, but fixed, number $c \in [-a,a]$ we can write

$$\zeta^{(k)}(t+a) = \zeta^{(k)}(t+c) + \int_c^a \frac{(a-s)^{m-k-1}}{(m-k-1)!}\, \zeta^{(m)}(t+s) ds,$$

$$k = 0,1,\ldots,m-1,$$

and $\zeta^{(k)}(t-a)$ can be expressed similarly. Then (4.4) can be rewritten as

$$\zeta^{(m)}(t + a) + b_m\zeta^{(m)}(t - a) + \sum_{k=0}^{m-1} b_k\zeta^{(k)}(t + c)$$

$$+ \int_{-a}^{a} \zeta^{(m)}(t + s)d\, v(s) = u(t), \tag{4.5}$$

where we have divided by $c_m \neq 0$ and have absorbed the functions $\tilde{v}(s)$,

$$\frac{(a - s)^{m-k-1}}{(m - k - 1)!}, \quad \frac{(-a - s)^{m-k-1}}{(m - k - 1)!}, \quad k = 0,1,\ldots, m - 1,$$

into the function of bounded variation $v(s)$ which now satisfies (4.3) with v_m replaced by v.

Finally, we define a function of two variables, $z(t,s)$, by

$$z(t,s) = \zeta(t + s), \; t \in (-\infty, \infty), \; \zeta \in [-a,a],$$

and note that the functional equation (4.5) is equivalent to the first order partial differential equation

$$\frac{\partial z^{(m)}(t,s)}{\partial t} = \frac{\partial z^{(m)}(t,s)}{\partial s}, \; t \in (-\infty, \infty) \; s \in (-a,a), \tag{4.6}$$

with the boundary conditions

$$z^{(m)}(t,a) + b_m z^{(m)}(t,-a) + \sum_{k=0}^{m-1} b_k z^{(k)}(t,c)$$

$$+ \int_{-a}^{a} z^{(m)}(t,s)\, dv(s) = u(t), \tag{4.7}$$

together with the differential equations

$$\frac{dz^{(k)}(t,c)}{dt} = z^{(k+1)}(t,c), \; k = 0,1,\ldots, m-1. \tag{4.8}$$

In (4.6), (4.7), (4.8), $z^{(k)}(t,s) = \zeta^{(k)}(t+s)$.

The form (4.6), (4.7), (4.8) of (4.5) is the easiest to use when formulating the system in the semigroup framework. It has the form

$$\frac{d\hat{z}}{dt} = A\hat{z} + \hat{b}u, \quad \hat{z} \in H^{m}[-a,a]$$

where

$$\hat{z}(t) = (z(t,c),\ldots, z^{(m-1)}(t,c), z^{(m)}(t,1)),$$

A is the operator defined by

$$A\hat{z} = (z'(c),\ldots, z^{(m)}(c), z^{(m+1)}(.))$$

on the domain consisting of $\hat{z} \in H^{m+1}[-a,a]$ for which the homogeneous counterpart of (4.7) holds, and $\hat{b}$ is the admissible control distribution element such that

$$\langle \hat{\omega},\hat{b}\rangle = \langle (\hat{\omega}(c),\ldots, \hat{\omega}^{(m-1)}(c), \hat{\omega}^{(m)}(.)), \hat{b}\rangle = \hat{\omega}^{(m)}(b)$$

for each $\hat{\omega}$ in the domain of $A^{*} \subseteq H^{m+1}[-a,a]$. (We shall have more to say about A^{*} and $\hat{b}$ for special instances of (4.6),(4.7),(4.8) in the next section.)

By making careful use of the condition (4.3) one can construct solutions of (4.6), (4.7), (4.8) by the method of successive approximations. The method already appears in essence in [9]. Not wishing to lengthen our presentation further, we content ourselves with the statement of

Theorem 4.1 The operator A generates a strongly continuous group, e^{At}, of bounded operators on $H^m[-a,a]$. For each $z_0(.) \epsilon H^m[-a,a]$

$$z(t,.) \equiv e^{tA} z_o(.)$$

is a generalized solution of (4.6), (4.7), (4.8) when $u(t) \equiv 0$ which is a classical solution if $z_0 \epsilon \mathrm{Dom}(A)$. If $u \epsilon L^2_{loc}(-\infty,\infty)$ then (4.6), (4.7), (4.8) has the generalized solution expressible in the form

$$z(t,.) = e^{tA} z_0(.) + K(t)u_t, K(t):L^2[0,t] \to H^m[-a,a],$$

where u_t denotes the restriction of u to $[0,t]$. (We replace $[0,t]$ by $[t,0]$ if $t < 0$.) Given $T > 0$, for $|t| \leq T$,

$$\| K(t)u_t \|_{H^m[a,b[} \leq M(T) \| u_t \|_{L^2[o,t]}$$

and if $|t_1| \leq T$, $|t_2| \leq T$, w.l.o.g. $t_1 \leq t_2$, then

$$\|K(t_1)u_{t_1} - K(t_2)u_{t_2}\|_{H^m[a,b]} \leq M(T)\| u_{(t_1,t_2)}\|_{L^2[0,t]}$$

where $M(T)$ is a positive number depending only on T and $u_{(t_1,t_2)}$ is the

restriction of u to $[t_1,t_2]$.

Now let us consider, as briefly as we can, the eigenvalue-eigenvector problem for the operator A. We may take $a = 1$ without loss of generality and it is convenient to take $c = 0$. Further, throughout this paper we will specialize to the case where

$$\int_{-1}^{1} \zeta(s)\, dv(s) = \int_{-1}^{1} \zeta(s)\, \hat{v}(s)ds, \quad \hat{v} \in L^2[-1,1].$$

More general cases will be treated elsewhere. Thus (4.7) assumes the form

$$z^{(m)}(t,1) + b_m z^{(m)}(t,-1) + \sum_{k=0}^{m-1} b_k\, z^{(k)}(t,0)$$

$$+ \int_{-1}^{1} z^{(m)}(t,s)\, \hat{v}(s)ds = u(t). \tag{4.9}$$

From (4.6) an eigenfunction of A takes the form $e^{\lambda_j s}$ where λ_j is a zero of the entire function $\Phi(\lambda)$ defined by

$$\Phi(\lambda) = \lambda^m e^{\lambda} + b_m \lambda^m e^{-\lambda} + \sum_{k=0}^{m-1} b_k\, \lambda^k + \lambda^m \int_{-1}^{1} e^{\lambda s}\hat{v}(s)ds. \tag{4.10}$$

Generalized eigenfunctions occur in groups $e^{\lambda_j s}$, $se^{\lambda_j s}$, ..., $(s^{\mu_j - 1}/(\mu_j - 1)!)e^{\lambda_j s}$ corresponding to possible multiple zeros of $\Phi(\lambda)$. We do not discuss this here, saving discussion of non-simple cases until Section 7.

Theorem 4.2 *The zeros of* $\Phi(\lambda)$, *defined by* (4.10), *consist, in the simple case, of* n *complex numbers* $\sigma_1, \sigma_2, \ldots, \sigma_m$ *together with a sequence* $\{\lambda_j \mid -\infty < j < \infty\}$:

$$\lambda_j = \alpha + j\pi i + \varepsilon_j, \tag{4.11}$$

where

$$\alpha = \tfrac{1}{2}\log(-b_m)$$

and

$$\sum_{j=-\infty}^{\infty} |\varepsilon_j|^2 < \infty .$$

Consequently, the functions $e^{\sigma_k t}$, $k = 1,2,\dots, m$, $e^{\lambda_j t}$, $-\infty < j < \infty$, *form uniform basis for* $H^m[-1,1]$.

Proof The classical Paley-Wiener-Levinson-Schwartz results ([12],[14],[19],[24]) show the $e^{\lambda_j t}$, $-\infty < j < \infty$, to form a uniform basis for $L^2[-1,1]$. The last stated result then follows from Theorem 3.1.

We proceed to give, as briefly as possible, the proof that the eigenvalues have the indicated form. Other arguments of this type are very common in the literature. (See, e.g. [2])

Let $\lambda = i\omega + \frac{1}{2}\log(-b_m) \equiv i\omega + \alpha$. Then, with $(-b_m)^{\frac{1}{2}}$ chosen as $-i\,b_m^{\frac{1}{2}}$, $0 \le \arg b_m^{\frac{1}{2}} < \pi$,

$$\Phi(\lambda) = \Psi(\omega) = 2b_m^{\frac{1}{2}}(i\omega + \alpha)^m \sin\omega + \sum_{k=0}^{m-1} b_k (i\omega + \alpha)^k$$

$$+ (i\omega + \alpha)^m \int_{-1}^{1} e^{(i\omega+\alpha)s}\, \hat{v}(s)\, ds$$

$$\equiv \psi_0(\omega) + \tilde{\psi}(\omega), \quad \psi_0(\omega) = 2b_m^{\frac{1}{2}}(i\omega + \alpha)^m \sin\omega. \tag{4.12}$$

Let Γ_k, $k = 1,2,3,\ldots,$ be the retangular contour whose sides are segments of $\mathrm{Re}(\omega) = \pm (k - \frac{1}{2})\pi$, $\mathrm{Im}(\omega) = \pm \beta$, where β is a fixed positive number soon to be specified. It is clear that there is a β_0 such that when $\beta > \beta_0$ and k is sufficiently large, $\psi_0(\omega)$ has $m + 2k - 1$ zeros inside Γ_k. Rouché's theorem gives the same conclusion for $\psi_0(\omega) + \tilde{\psi}(\omega)$ provided we can show that

$$\left| \frac{\tilde{\psi}(\omega)}{\psi_0(\omega)} \right| < 1, \quad \omega \in \Gamma_k, \tag{4.13}$$

for β chosen appropriately large and k sufficiently large. Now

$$\frac{\tilde{\psi}(\omega)}{\psi_0(\omega)} = \frac{b_m^{-\frac{1}{2}}}{2} \left[\sum_{k=0}^{m-1} b_k (i\omega + \alpha)^{k-m} \csc\omega \right.$$

$$\left. + \int_{-1}^{1} e^{(i\omega+\alpha)s} \csc\omega \, \hat{v}(s)\, ds \right]$$

Since $k - m$ is a negative integer and since $\csc\omega \to 0$ as $|\mathrm{Im}\,(\omega)| \to \infty$, we can make the first term as small as we wish on Γ_k, for sufficiently large k, by choosing β appropriately large.

On the vertical sides of Γ_k we have $\omega = \pm (k - \frac{1}{2})\pi + i\sigma$, $|\sigma| \leq \beta$, so

$$\left| \int_{-1}^{1} e^{(i\omega+\alpha)s} \csc\omega \, \hat{v}(s) ds \right| = 2 \left| \int_{-1}^{1} \frac{e^{\pm(k-\frac{1}{2})\pi i s} e^{(\alpha-\sigma)s} \hat{v}(s) ds}{e^{\pm i(k-\frac{1}{2})\pi} e^{-\sigma} - e^{\mp i(k-\frac{1}{2})\pi} e^{\sigma}} \right| \tag{4.14}$$

and on the horizontal sides we have $\omega = \rho \pm \beta i$, $-(k - \frac{1}{2})\pi \leq \rho \leq (k - \frac{1}{2})\pi$, so

$$\left| \int_{-1}^{1} e^{(i\omega+\alpha)s} \csc\omega \, \hat{v}(s) ds \right| = 2 \int_{-1}^{1} \left| \frac{e^{i\rho s} e^{(\alpha\mp\beta)s} \hat{v}(s)\, ds}{e^{i\rho} e^{\mp\beta} - e^{-i\rho} e^{\pm\beta}} \right| \tag{4.15}$$

On the right hand of (4.14) the denominator is bounded away from zero for all k and the basic argument of the Riemann-Lebesque lemma is used to show that the integral becomes small as $|k| \to \infty$, uniformly for $\omega = \pm (k - \frac{1}{2})\pi + i\sigma$, $|\sigma| \leq \beta$. On the right hand side of (4.15) the integrand is bounded in $L^2[-1,1]$, hence in $L^1[-1,1]$, uniformly for $\omega = \rho \pm i\beta$, $-k(-\frac{1}{2})\pi \leq \rho \leq (k - \frac{1}{2})\pi$, and we obtain the result from the fact that the modulus of the denominator may be made as large as we wish, uniformly for such ω, by taking β sufficiently large.

Having established (4.12), and hence that $\psi_0(\omega) + \tilde{\psi}(\omega)$ has $m + 2k - 1$ zeros inside Γ_k for k sufficiently large, we proceed to obtain the asymptotic formula (4.11). We regroup the terms in (4.12):

$$\psi_0(\omega) + \tilde{\psi}(\omega) = \theta_k(\omega) + \tilde{\theta}_k(\omega),$$

$$\theta_k(\omega) = 2b_m^{\frac{1}{2}} (i\omega + \alpha)^m \sin \omega + (i\omega + \alpha)^m \frac{b_{m-1}}{(ik\pi + \alpha)} + (i\omega + \alpha)^m \int_{-1}^{1} e^{ik\pi s} e^{s} \hat{v}(s)ds$$

$$\tilde{\theta}_k(\omega) = \sum_{k=0}^{m-2} b_k (i\omega + \alpha)^k + (i\omega + \alpha)^m \left(\frac{b_{m-1}}{i\omega + \alpha} - \frac{b_{m-1}}{ik\pi + \alpha} \right)$$

$$+ (i\omega + \alpha)^m \int_{-1}^{1} \left(e^{(i\omega + \alpha)s} - e^{(ik\pi + \alpha)s} \right) \hat{v}(s)ds.$$

Now the numbers

$$d_k = \frac{b_{m-1}}{ik\pi + \alpha},$$

$$e_k = \int_{-1}^{1} e^{ik\pi s} e^{s} \hat{v}(s)ds,$$

(the e_k being Fourier coefficients of $e^{\alpha s}\hat{v}(s)$) are square summable. Then, for $|k|$ sufficiently large, $\theta_k(\omega)$ has a zero at $\tilde{\omega}_k$, where

$$2b_m^{\frac{1}{2}} \sin(\tilde{\omega}_k) + d_k + e_k = 0$$

i.e., at

$$\tilde{\omega}_k = k\pi - (-1)^k \arcsin\left(\frac{d_k + e_k}{2b_m^{\frac{1}{2}}}\right) \equiv k\pi + \tilde{\delta}_k .$$

Clearly the $\tilde{\delta}_k$ are square summable. Now

$$\frac{\tilde{\theta}_k(\omega)}{\theta_k(\omega)} = \frac{1}{2b_m^{\frac{1}{2}} \sin \omega + d_k + e_k} \left\{ \sum_{k=0}^{m-2} b_k (i\omega + \alpha)^{k-m} \right.$$

$$\left. + \frac{ib_{m-1}(k\pi - \omega)}{(i\omega + \alpha)(ik\pi + \alpha)} + \int_{k\pi}^{\omega} \int_{-1}^{1} ise^{iws} e^{\alpha s} \hat{v}(s)\,ds\,dw \right\} ,$$

the w integration taking place over the *straight line segment* L_k joining ω to k .

Let C_k be a circle of radius

$$r_k = 2|\delta_k| + \frac{1}{|k\pi|}$$

centered at $k\pi$.

Then it is not hard to see that

$$\left| \frac{1}{2b_m^{\frac{1}{2}} \sin \omega + d_k + e_k} \right| = \mathcal{O}\left(\frac{1}{|\delta k| + \frac{1}{|k\pi|}} \right), \quad |k| \to \infty ,$$

uniformly for $\omega \in C_k$. But clearly

$$\left| \sum_{k=0}^{m-2} b_k \, (i\omega + \alpha)^{k-m} \right| = \mathcal{O}\left(\frac{1}{k^2 \pi^2}\right), \quad |k| \to \infty ,$$

uniformly for $\omega \in C_k$. Further, again using a variant of the Riemann-Lebesque lemma, one can show that

$$\int_{-1}^{1} e^{iws} \left[ise^{\alpha s} \hat{v}(s) \right] ds$$

tends uniformly to zero for $w \in L_k$ as $|k|$ tends to infinity. Thus

$$\left| \int_{k\pi}^{\omega} \int_{-1}^{1} ise^{iws} \, e^{\alpha s} \, \hat{v}(s) ds \, dw \right| = \mathcal{O}(|\omega - k\pi|) = \mathcal{O}\left(2|\delta_k| + \frac{1}{|k\pi|}\right),$$

$$|k| \to \infty .$$

Then

$$\left| \frac{1}{2b_m^{\frac{1}{2}} \sin \omega + d_k + e_k} \int_{k\pi}^{\omega} \int_{-1}^{1} ise^{iws} \, e^{\alpha s} \, \hat{v}(s) ds \, dw \right|$$

$$= \mathcal{O}\left(\frac{1}{|\delta_k| + \frac{1}{|k\pi|}} \right) \mathcal{O}\left(2|\delta_k| + \frac{1}{k\pi} \right) = \mathcal{O}(1), \quad |k| \to \infty .$$

We conclude that

$$\lim_{|k| \to \infty} \left| \frac{\tilde{\theta}_k(\omega)}{\theta_k(\omega)} \right| = 0$$

and Rouche's theorem tells us that $\psi_o(\omega) + \tilde{\psi}(\omega)$ has a **zero** ω_k inside C_k for

sufficiently large k. We have, clearly,

$$\omega_k = k\pi + \mathcal{O}\left(2|\delta_k| + \frac{1}{|k\pi|}\right), \quad |k| \to \infty ,$$

Then $\psi(\lambda)$ has, for sufficiently large k, a zero

$$\lambda_k = \tfrac{1}{2}\log(-b_m) + i\omega_k \tag{4.16}$$

$$= \tfrac{1}{2}\log(-b_m) + k\pi i + \mathcal{O}\left(2|\delta_k| + \frac{1}{|k\pi|}\right) .$$

Combined with the result conccerning the number of zeros inside the contour Γ_k, the asymptotic formula (4.16) completes the proof of the theorem.

We complete this section with a short discussion of neutral functional equations of negative order $-m$, $m = 1,2,3,\ldots$. We have seen that neutral equations of non-negative order m with delay interval 2, have as their basic state spaces the spaces $H^m[-1,1]$ and the corresponding exponential eigenfunctions of the operator A form uniform bases for those spaces. In view of Theorem 3.2 it is perhaps not surprizing that there should also be a family of neutral functional equations having the spaces $H^{-m}[-1,1]$ as their basic state spaces.

Definition 4.3 *A neutral functional equation of order* $-m$, $m = 1,2,3,\ldots,$ *is an equation of the form*

$$\int_{-1}^{1} \overline{v(-s)}\zeta(t + s)\, ds = u(t) \tag{4.17}$$

where $v \in H^m[-1,1]$ *is such that*

$$v^{(m-1)}(1) \neq 0,\ v^{(m-1)}(-1) \neq 0, \tag{4.18}$$

$$v^{(j)}(1) = v^{j}(-1) = 0,\ j = 0,1,\ldots,\ m-2. \tag{4.19}$$

It is possible to extend this definition to the case where $v \in H^m[s_{n-1}, s_n]$, $n = 0,1,2,\ldots, N$ where $-1 = s_0 < s_1 < s_2 < \ldots s_N = 1$, (4.18), (4.19) hold at s_0, s_N, $v^{(m-1)}$ has discontinuities at s_n, $n = 1,2,\ldots, N-1$ but $v^{(j)}$ is continuous there, $j = 0,1,2,\ldots, m-2$. We hope to have more to say about this elsewhere. Clearly (4.19) is vacuous for $m = 1$.

Of course we must demonstrate that (4.17) has solutions in some acceptable sense. We will do that here by what might be termed the method of separation of variables. There are many other possibilities.

Theorem 4.4 Consider the operator

$$(Az)(s) = z'(s)$$

in $H^{-m}[-1,1]$, defined on the domain consisting of elements $z \in H^{-m+1}[-1,1]$ for which

$$\langle z,v\rangle = 0.$$

(Here $\langle \zeta,v\rangle$ is the extension of the linear functional (4.17), defined in terms of $v \in H_0^{m-1}[-1,1]$, to a linear functional on $H^{-m+1}[-1,1]$.) Then A has (in the simple case) exponential eigenfunctions $e^{\lambda_j s}$, $j \in J_m$, where J_m is a subset of $\{j \mid -\infty < j < \infty\} \equiv J$ which is obtained by removing m elements of J. The complex numbers λ_j have the form

$$\lambda_j = \alpha + j\pi i + \varepsilon_j, \; j \in J_m, \tag{4.20}$$

$$\sum_{j \in J_m} |\varepsilon_j|^2 < \infty . \tag{4.21}$$

Letting $\sigma_1, \sigma_2, \ldots, \sigma_m$ be m distinct complex numbers not equal to any λ_j and letting $p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k)$, these exponential eigenfunctions are such that $p(\lambda_j) e^{\lambda_j s}$, $j \in J_m$, form a uniform basis for $H^{-m}[-1,1]$.

Remarks Strictly speaking, we should include the possibility that A has finitely many eigenvalues of multiplicity greater than 1. We consider only the simple case here, postponing all discussion of multiple eigenvalues to Section 7.

The integral expression

$$\int_{-1}^{1} \overline{v(-s)}\, \zeta(s)\, ds, \tag{4.22}$$

defined initially for $\zeta \in L^2[-1,1]$, may be seen to be continuous with respect to the $\|\ \|_{-m+1}$ norm of ζ. Let $\eta \in H^{m-1}[-1,1]$ be defined, for $\zeta \in L^2[-1,1]$, by

$$\frac{d^{m-1}\eta}{ds^{m-1}} = \zeta, \; \eta^{(j)}(0) = 0, \; j = 0, \ldots, m-2.$$

Then $\|\zeta\|_{-m+1}$ is equivalent to $\|\eta\|_{L^2[-1,1]}$ and

$$\left| \int_{-1}^{1} v(-s)\, \zeta(s) ds \right| = \left| \int_{-1}^{1} v^{(m-1)}(-s)\, \eta(s) ds \right|$$

$$\leq K \|v\|_{m-1} \|\eta\|_{L^2[-1,1]} \leq \hat{K} \|v\|_{m-1} \|\zeta\|_{-m+1}$$

for suitable positive constants K, $\hat{K}$. Accordingly, (4.20) extends to a linear functional on $H^{-m-1}[-1,1]$ which we denote by $\langle \zeta, v\rangle$ for ζ in that space. Thus (4.17) can also be written as $\langle \zeta(t+\cdot), v\rangle = u(t)$. With $z(t,s) = \zeta(t+s)$, our system is

$$\frac{\partial z(t,s)}{\partial t} = \frac{\partial z(t,s)}{\partial s} \tag{4.23}$$

$$\langle z(t,\cdot),\ v\rangle = u(t) \tag{4.24}$$

Proof of Theorem 4.4 The last statement is clear, given (4.20),(4.21), since the functions $e^{\sigma_k s}$, $k = 1,2,\ldots, m$, $e^{\lambda_j s}$, $j \in J_m$, form uniform basis for $H^0[-1,1] = L^2[-1,1]$ and we have Theorem 3.2. Thus we need only establish the asymptotic form of the λ_j, (4.20), together with the indexing restriction, $j \in J_m$.

The eigenfunctions are clearly exponentials, the exponents determined by (4.24) with $u(t) \equiv 0$. Consider the entire function

$$\theta(\lambda) = \int_{-1}^{1} \overline{v(-s)}\, e^{\lambda s}\, ds$$

Since $v(s) \not\equiv 0$, $\theta(\lambda) \not\equiv 0$, and we may select m complex numbers, the $\sigma_1, \sigma_2, \ldots \sigma_m$ referred to in the theorem statement, such that

$$\theta(\sigma_k) \neq 0,\ k = 1,2,\ldots, m.$$

Let

$$\psi(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k)\theta(\lambda) .$$

Then it is easy to see that

$$\psi(\lambda) = \int_{-1}^{1} \overline{v(-s)} \; L e^{\lambda s} \, ds$$

where

$$L\zeta = \prod_{k=1}^{m} (D - \sigma_k I)\zeta, \quad \bar{L}\zeta = \prod_{k=1}^{m} (D - \bar{\sigma}_k I)\zeta.$$

But, integrating by parts and using (4.18), (4.19),

$$\psi(\lambda) = \int_{-1}^{1} \overline{(\bar{L}v)(-s)} \; e^{\lambda s} \, ds + \overline{v^{(m-1)}}(-1) e^{\lambda} - \overline{v^{(m-1)}}(1) \; e^{-\lambda}$$

$$= \int_{-1}^{1} \overline{h(s)} \; e^{\lambda s} \, ds + \overline{v^{(m-1)}} \; (-1) \; e^{\lambda} - \overline{v^{(m-1)}} \; (1) \; e^{-\lambda} ,$$

where $h \in L^2[-1,1]$. Proceeding as in the proof of Theorem 4.2 in the case $m = 0$, we conclude that the zeros of $\psi(\lambda)$ have the form (4.20), (4.21) with

$$\alpha = \tfrac{1}{2} \log \left(\frac{v^{(m-1)}(1)}{v^{(m-1)}(-1)} \right)$$

But the zeros of $\psi(\lambda)$ include $\sigma_1, \sigma_2, \ldots, \sigma_m$, which are not included in the zeros of $\Theta(\lambda)$. It is easy then to see that A has eigenfunctions $e^{\lambda_j s}$, $j \in J_m$ as described in the theorem, the λ_j, $j \in J_m$, being the zeros of $\psi(\lambda)$ other than $\sigma_1, \sigma_2, \ldots, \sigma_m$.

By virtue of the uniform basis property of the $p(\lambda_j) e^{\lambda_j t}$, $j \in J_m$, in $H^{-m}[-1,1]$, solutions of the homogeneous system ((4.23),(4.24) with $u(t) \equiv 0$) can be written as

$$z(t,s) = \sum_{j \in J_m} z_j(t) p(\lambda_j) e^{\lambda_j s} \tag{4.25}$$

with

$$\sum_{j\epsilon J_m} |z_j(0)|^2 < \infty,$$

$$\frac{dz_j}{dt} = \lambda_j z_j, \; j \epsilon j_m.$$

Thus the $z_j(t) = e^{\lambda_j t} z_j(0)$ also satisfy, using (4.20), (4.21),

$$\sum_{j \epsilon J_m} |z_j(t)|^2 < \infty$$

So $z(t,\cdot)$ lies in $H^{-m}[-1,1]$ for all t. The *strong continuity* is readily established.

There remains the problem of treating the inhomogeneous problem adequately. We can use Theorem 1.3 here, provided we can identify the control distribution element b appropriate to the system (4.23), (4.24) and estimate the control distribution coefficients.

Let us proceed formally for the moment. If

$$z(t,s) = \sum_{j\epsilon J_m} z_j(t)\, p(\lambda_j)\, e^{\lambda_j s}$$

is a solution of (4.23), (4.24), then

$$z(t,s) = L\eta(t,s), \quad L = \prod_{k=1}^{m} \left(\frac{\partial}{\partial s} - \sigma_k I\right),$$

where

$$\eta(t,s) = \sum_{j\epsilon J_m} z_j(t)\, e^{\lambda_j s} + \sum_{k=1}^{m} \tilde{z}_k(t)\, e^{\sigma_k s}$$

for any choice of the $\tilde{z}_k(t)$. Then we have, in place of (4.24)

$$\langle L\eta, v\rangle = u(t)$$

which can be transformed, as before, to

$$\frac{\partial\eta(t,s)}{\partial t} = \frac{\partial\eta(t,s)}{\partial s} \tag{4.26}$$

$$\overline{v^{(m-1)}(-1)}\,\eta(t,1) - \overline{v^{(m-1)}(1)}\,\eta(t,1) + \int_{-1}^{1} \overline{h(s)}\,\eta(t,s)ds = u(t) \tag{4.27}$$

provided

$$\frac{dz_j}{dt} = \lambda_j z_j + b_j\, u(t),\ j \in J_m,$$

$$\frac{d\tilde{z}_k(t)}{dt} = \sigma_k \tilde{z}_k(t) + \tilde{b}_k\, u(t),$$

where the b_j, $\tilde{b}_k$ are the control distribution coefficients for (4.26), (4.27). We will establish in Section 5 that the b_j, $\tilde{b}_k$ are uniformly bounded and uniformly bounded away from zero. Then the $z_j(t)$ satisfy

$$\frac{dz_j(t)}{dt} = \lambda_j z_j(t) + b_j\, u(t) \tag{4.28}$$

so that the control distribution coefficients for (4.23), (4.24) are just the numbers b_j, $j \in J_m$, which are thus bounded and bounded away from zero. Then Theorem 1.3 may be applied to (4.26), (4.27) to show that there are solutions of that system and they have, in $H^{-m}[-1,1]$, the properties given in Theorem 1.3.

5. Bounds on the Control Distribution Coefficients

We will study the system (4.4) in the form (4.6), (4.7) (taking, without loss of generality, $a = 1$, $c = 0$),(4.9). Thus we have

$$\frac{\partial z^{(m)}(t,s)}{\partial t} = \frac{\partial z^{(m)}(t,s)}{\partial s}, \quad -1 < s < 1, \; -\infty < t < \infty, \tag{5.1}$$

$$\frac{dz^{(k)}(t,0)}{dt} = z^{(k+1)}(t,0), \quad k = 0,1,\ldots, m-1, \tag{5.2}$$

$$z^{(m)}(t,1) + b_m z^{(m)}(t,-1) + \sum_{k=0}^{m-1} b_k z^{(k)}(t,0) + \int_{-1}^{1} \hat{v}(s) z^{(m)}(t,s)ds$$

$$= u(t), \quad \hat{v} \in L^2[-1,1]. \tag{5.3}$$

We wish to develop this system in the form (1.10),(1.11). Since we treat only the simple case at present, $\Lambda_j = \lambda_j$ (or σ_k) will be scalar. Taking account of Theorem 4.2, for m non-negative the relevant from should be

$$\frac{d\tilde{z}_k}{dt} = \sigma_k \tilde{z}_k + \tilde{b}_k u, \; k = 1,2,\ldots, m, \quad \frac{dz_j}{dt} = \lambda_j z_j + b_j u,$$

$$-\infty < j < \infty, \tag{5.4}$$

for certain control distribution coefficients b_j, $\tilde{b}_k$. It will be important to demonstrate that these actually exist and to obtain bounds on them. This requires the introduction of a system adjoint to (5.1),(5.2),(5.3) in an appropriate sense.

Let $(\;,\;)_m$ denote the inner product in $H^m[-1,1]$ as defined in (3.3), let $z(t,s) = (z(t,0), z'(t,0)\ldots, z^{m-1}(t,0), z^{(m)}(t,s))$ be a solution

of (5.1), (5.2), (5.3), initially with $u(t) \equiv 0$ and $z(0,s) \equiv z_0(s)$ in the domain of the operator A, so that solutions may be differentiated without qualm. Letting $w(t,s)$ be a smooth function of t,s, we compute that

$$\frac{d}{dt}(z(t,.), w(t,.))_m = z(t,0)\ \overline{\frac{dw(t,0)}{dt}}$$

$$+ \sum_{k=0}^{m-2} z^{(k+1)}(t,0)\ \overline{\left[w^{(k)}(t,0) + \frac{dw^{(k+1)}(t,0)}{dt}\right]} + z^{(m)}(t,0)\ \overline{w^{(m-1)}(t,0)}$$

$$+ \int_{-1}^{1} \left[\frac{\partial z^{(m)}(t,s)}{\partial s}\ \overline{w^{(m)}(t,s)} + z^{(m)}(t,s)\ \overline{\frac{\partial w^{(m)}(t,s)}{\partial t}}\right] ds. \quad (5.5)$$

Noting that for $\sigma \in [-1,1]$

$$z^{(m)}(t,0) = z^{(m)}(t,\sigma) - \int_0^{\sigma} \frac{\partial z^{(m)}(t,s)}{\partial s}\, ds ,$$

we have

$$z^{(m)}(t,0) = \tfrac{1}{2}\int_{-1}^{1} z^{(m)}(t,s)\, ds - \tfrac{1}{2}\int_{-1}^{1}\int_0^{\sigma} \frac{\partial z^{(m)}(t,s)}{\partial s}\, ds\, d\sigma$$

$$= \int_{-1}^{1} z^{(m)}(t,s)\, ds - \tfrac{1}{2}\int_{-1}^{1} (1 - |s|)\, \frac{\partial z^{m}(t,s)}{\partial s}\, ds. \quad (5.6)$$

Substituting (5.6) into (5.5), integrating by parts and then using the boundary conditions (5.3), we obtain

$$\frac{d}{dt}(z(t,.),w(t,.))_m = z(t,0)\overline{\left[\frac{dw(t,0)}{dt} - \overline{b_0}\, w^{(m)}(t,1)\right]}$$

$$+ \sum_{k=0}^{m-2} z^{(k+1)}(t,0)\,\overline{\left[w^{(k)}(t,0) + \frac{dw^{(k+1)}(t,0)}{dt} - \overline{b_{k+1}}\, w^{m}(t,1)\right]}$$

$$- z^{(m)}(t,-1)\left[\overline{b_m}\, w^{(m)}(t,1) + w^{(m)}(t,-1)\right]$$

$$+ \int_{-1}^{1} z^{(m)}(t,s)\,\overline{\left[\frac{\partial w^{(m)}(t,s)}{\partial t} - \frac{\partial w^{(m)}(t,s)}{\partial s} + \tfrac{1}{2}(1 - \mathrm{sgn}(s)w^{(m-1)}(t,0)\right.}$$

$$\overline{\left.- \overline{\hat{v}(s)}\, w^{(m)}(t,1)\right]}\, ds.$$

The derivative of the inner product is thus equal to zero if $w(t,s)$ satisfies the adjoint system ($sgn(s) = 1,\ s > 0,\ = -1,\ s < 0$)

$$\frac{\partial w^{(m)}(t,s)}{\partial t} = \frac{\partial w^{(m)}(t,s)}{\partial s} - \tfrac{1}{2}(1 - \mathrm{sgn}(s))w^{(m-1)}(t,0) - \overline{\hat{v}(s)}\, w^{(m)}(t,1) \tag{5.7}$$

$$\frac{dw^{(k)}(t,0)}{dt} = \overline{b_0}\, w^{(m)}(t,1)$$

$$\frac{dw^{(k)}(t,0)}{dt} = \overline{b_k}\ \ w^{(m)}(t,1) - w^{(k-1)}(t,0),\ k = 1,2,\ldots,m-1, \tag{5.8}$$

$$\overline{b_m}\, w^{(m)}\ (t,1) + w^{(m)}(t,-1) = 0. \tag{5.9}$$

When $u(t)$ is not identically zero and (5.1), (5.2), (5.3), (5.7), (5.8), (5.9) hold, we do not obtain

$$\frac{d}{dt}\,(z(t,),\ w(t,.))_m = 0,$$

but rather

$$\frac{d}{dt}\,(z(t,.),\ w(t,.)) = \overline{w^{(m)}(t,1)\ u(t)}. \tag{5.10}$$

With A defined as in (4.8) ff. ($a = 1$, $c = 0$), differentiable solutions of (5.7), (5.8), (5.9) have the form $w(t,.) = S(T - t)^*\psi$, ψ in the domain of A^*. Taking (1.8) ff. into account, the relevant admissible control distribution element for (5.1), (5.2), (5.3) is the linear functional b defined for w in the domain of A^* (the subspace of $H^{m+1}[-1,1]$ defined by (5.9)) by

$$\langle w,b\rangle = w^{(m)}(1). \tag{5.11}$$

With this b, (5.1), (5.2), (5.3) can be written

$$\frac{dz}{dt} = Az + bu. \tag{5.12}$$

The corresponding form of (5.7), (5.8), (5.9) is $\frac{dw}{dt} = -A^*w$. The control distribution coefficients are (cf. (1.7))

$$\tilde{b}_k = \langle \overline{\tilde{q}_k}, b\rangle = \overline{\tilde{q}_k^{(m)}(1)},\ k = 1,2,\ldots, m, \tag{5.13}$$

$$b_j = \langle \overline{q_j}, b\rangle = \overline{q_j^{(m)}}(1)\quad -\infty < j < \infty, \tag{5.14}$$

where $\tilde{q}_k$, q_j are the elements of the dual basis relative to the functions

$e^{\sigma_k t}$, $e^{\lambda_j t}/p(\lambda_j)$ in $H^m[-1,1]$. We are now able to prove

<u>Theorem 5.1</u> <u>There are positive numbers</u> M_1, M_2 <u>such that</u>

$$0 < M_1 \le |\tilde{b}_k| \le M_2, \quad k = 1,2,\dots, m, \tag{5.15}$$

$$M_1 \le |b_j| \le M_2, \quad -\infty < j < \infty. \tag{5.16}$$

<u>Proof</u> The $\tilde{q}_k$, q_j are biorthogonal to the $e^{\sigma_k t}$, $e^{\sigma_j t}$ in the sense that

$$\begin{pmatrix} (e^{\sigma_k t}, \tilde{q}_{\hat{k}})_m, & (e^{\sigma_k t}, q_{\hat{j}})_m \\ \left(\dfrac{e^{\lambda_j t}}{p(\lambda_j)}, q_{\hat{k}}\right)_m, & \left(\dfrac{e^{\lambda_j t}}{p(\lambda_j)}, q_{\hat{j}}\right)_m \end{pmatrix} = \begin{pmatrix} \delta_{k,\hat{k}} & 0 \\ 0 & \delta_{j,\hat{j}} \end{pmatrix} .$$

Clearly it is enough to establish two things. First, that an inequality like (5.16) holds for $|j|$ sufficiently large, and, secondly, that no $\tilde{b}_k$ or b_j is equal to zero. In view of (5.13), (5.14), one need only prove these statements for the numbers $\tilde{q}_k^{(m)}(1)$, the complex function $\phi(\lambda)$ defined by (4.12), which, with $\lambda = i\omega$, can be written as

$$\phi(\lambda) = \hat{\psi}(\omega) = (i\omega)^m e^{i\omega} + b_m (i\omega)^m e^{-i\omega} + \sum_{k=0}^{m-1} b_k (i\omega)^k$$

$$+ (i\omega)^m \int_{-1}^{1} e^{i\omega s}\, \hat{v}(s)\, ds. \tag{5.17}$$

With $\lambda_j = i\omega_j$, $\sigma_k = i\nu_k$, we have $\hat{\psi}(\omega_j) = \hat{\psi}(\nu_k) = 0$, and we are assuming that this is a simple zero (this can be shown true for $|j|$ large enough). We define

$$\psi_j(\omega) = \frac{p(\lambda_j)\,\hat{\psi}(\omega)}{\hat{\psi}'(\omega_j)(\omega - \omega_j)}\,, \qquad \omega \neq \omega_j \tag{5.18}$$

$$= p(\lambda_j), \quad \omega = \omega_j. \tag{5.19}$$

Thus $\psi_j(\omega_j) = 1$, $\psi_j(\omega_{\hat{j}}) = 0$, $\hat{j} \neq j$, $\psi_j(\nu_k) = 0$. Now $\psi_j(\omega)$ has the following properties. It is an entire function of ω of order 1 and type 1 such that $\psi_j(\omega)/(1 + |\omega|^m)$ is square integrable on any line parallel to the real ω axis, and, using the easily established fact that the terms $p(\lambda_j)/\hat{\psi}'(\omega_j)$ are bounded, we have

$$|\psi_j(\omega)| \leq K(1 + |\omega|)^{m-1}\, e^{|Im(\omega)|}, \; -\infty < j < \infty,$$

for all complex ω. Expanding $\psi_j(\omega)$ in Taylor series about $\omega = 0$ we have

$$\psi_j(\omega) = \psi_j(\omega) + \psi_j'(0)\omega + \ldots + \psi_j^{(m-1)}(0)\,\frac{\omega^{m-1}}{(m-1)!} + \omega^m\,\theta_j(\omega), \tag{5.20}$$

where $\theta_j(\omega)$ is an entire function of ω and, from the corresponding properties of $\psi_j(\omega)$, $\theta_j(\omega)$ has the following properties: $\theta_j(\omega)$ is of order 1 and type 1 with $\theta_j(\omega)$ square integrable on any line parallel to the real ω axis, and

$$|\theta_j(\omega)| \leq \frac{K_1}{(1 + |\omega|)}\, e^{|Im(\omega)|}$$

for all complex ω. It follows [14] that $\theta_j(\omega)$ may be represented as

$$\theta_j(\omega) = i^m \int_{-1}^{1} e^{i\omega s} v_j^{(m)}(s)\, ds \tag{5.21}$$

with $v_j^{(m)} \in L^2[-1,1]$. Then $\psi_j(\omega)$ is the $H^m[-1,1]$ inner product of $e^{i\omega s}$ with the element

$$v_j = \left(\overline{\psi_j(0)},\ \overline{i\psi_j'(0)},\ \ldots\ \frac{\overline{i^{m-1}\ \psi_j^{(m-1)}(0)}}{(m-1)!},\ \overline{v_j^{(m)}(.)} \right)$$

in $H^m[-1,1]$. From the values of $\psi_j(\omega)$ at the points ν_j, ω_j we see that

$$(e^{\sigma_k t}, v_j)_m = 0,\ -\infty < j < \infty,\quad k = 1,2,\ldots, m$$

$$\left(\frac{e^{\lambda_j t}}{p(\lambda_j)},\ v_{\hat{j}} \right)_m = \delta_{j,\hat{j}} \quad -\infty < j,\hat{j} < \infty$$

and it follows from the uniqueness of the dual basis to the $e^{\sigma_k t}$, $\frac{e^{\lambda_j t}}{p(\lambda_j)}$ in $H^m[-1,1]$ that

$$q_j = v_j \tag{5.22}$$

Then

$$q_j^{(m)}(1) = v_j^{(m)}(1)\ .$$

Using (5.22) in (5.21) we infer that

$$\theta_j(\omega) = i^m \int_{-1}^{1} e^{i\omega s}\ \overline{q_j^{(j)}(s)}\ ds$$

so that

$$\int_{-1}^{1} e^{i\omega s}\ \overline{q_j^{(m)}(s)}\ ds = i^{-m}\ \theta_j(\omega)$$

$$= (\text{from } (5.20)) = \frac{1}{(i\omega)^m} \left[\psi_j(\omega) - \sum_{k=0}^{m-1} \frac{\psi_j^{(k)}(0)}{k!}\ \omega^k \right] \tag{5.23}$$

where $v \in H$ [-1,1] is such that

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Since q_j belongs to the domain of A^*, $q_j \in H^{m+1}[-1,1]$. Integrating by parts on the left hand side of (5.23) and using (5.18) on the right hand side we have

$$\overline{q_j^{(m)}(1)}\, e^{i\omega} - \overline{q_j^{(m)}(-1)}\, e^{-i\omega} - \int_{-1}^{1} e^{i\omega s}\, \overline{q_j^{(m+1)}(s)}\, ds$$

$$= \frac{1}{(i\omega)^{m-1}} \left[\frac{p(\lambda_j)}{\hat{\psi}'(\omega_j)(\omega - \omega_j)} \left((i\omega)^m e^{i\omega} + b_m (i\omega)^m e^{-i\omega} \right.\right.$$

$$\left.\left. + \sum_{k=0}^{m+1} b_k (i\omega)^k + (i\omega)^m \int_{-1}^{1} e^{i\omega s}\, \hat{v}(s)\, ds \right) - \sum_{k=0}^{m-1} \frac{\psi_j^{(k)}(0)}{k!}\, \omega^k \right] \tag{5.24}$$

Examining the terms on both sides, we conclude that

$$q_j^{(m)}(1) = \lim_{\mathrm{Im}(\omega) \to -\infty} \overline{\left(\frac{i\,\omega\, p(\lambda_j)}{\hat{\psi}'(\omega_j)(\omega - \omega_j)} \right)} = \overline{\frac{i\, p(\lambda_j)}{\hat{\psi}'(\omega_j)}}\,, \tag{5.25}$$

$$q_j^{(m)}(-1) = \lim_{\mathrm{Im}(\omega) \to +\infty} \overline{\left(\frac{-i\omega b_m p(\lambda_j)}{\hat{\psi}'(\omega_j)(\omega - \omega_j)} \right)}\,,$$

$$= \overline{\left(\frac{-i\, b_m\, p(\lambda_j)}{\hat{\psi}'(\omega_j)} \right)}\,. \tag{5.26}$$

This shows the $q_j^m(1)$ are not zero and it clearly may be seen to be in agreement with (5.9). Now it is an easy matter to verify from the formula (5.17) that there is a non-zero constant b such that

$$\lim_{|j| \to \infty} \frac{p(\lambda_j)}{\hat{\psi}'(\omega_j)} = \lim_{|j| \to \infty} \frac{p(i\omega_j)}{\hat{\psi}'(\omega_j)} = b. \tag{5.27}$$

To develop similar formulae for the $\tilde{q}_k^{(m)}(\pm 1)$ one simply replaces (5.18), (5.19) by

$$\tilde{\psi}_k(\omega) = \frac{\hat{\psi}(\omega)}{\hat{\psi}'(\nu_k)(\omega - \nu_k)}, \omega \neq \nu_k = 1, \omega = \nu_k .$$

and we obtain in place of (5.25), (5.26)

$$\hat{q}_k^{(m)}(1) = \frac{\overline{i}}{\hat{\psi}'(\nu_k)} \neq 0, \tag{5.28}$$

$$\hat{q}_k^{m}(-1) = \frac{\overline{i\, b_m}}{\hat{\psi}'(\nu_k)} \neq 0 . \tag{5.29}$$

Combining (5.25), (5.26) with (5.28), (5.29) and (5.27) the theorem is proved.

The above takes care of nonnegative values of m. For negative values of m the result just proved, in the case $m = 0$, together with the discussion at the end of Section 4, which depends on the result for $m = 0$ proved above, shows that the control distribution coefficients for the equation (4.17) also satisfy inequalities of the form (5.16) for $j \in J_m$.

$L^2[-1,1]$

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6. Control Canonical Forms and Spectral Assignment Theorems

In the system (2.12) - (2.16) of Section 2 we have an example of an augmented hyperbolic system

$$\dot{x} = Ay + gu, \; x, \; g \in H \tag{6.1}$$

wherein A has eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_m$, and (cf. (2.10) ff.)

$$\left.\begin{aligned} &\lambda_j = \alpha + j\pi i + \varepsilon_j, \; -\infty < j < \infty, \\ &\sum_{j=-\infty}^{\infty} |\varepsilon_j|^2 < \infty, \quad \alpha = \tfrac{1}{2}\log \alpha. \end{aligned}\right\} \tag{6.2}$$

Assuming that the corresponding eigenvectors ϕ_k, $k = 1,2,\ldots, m$, ϕ_j, $-\infty < j < \infty$, form a uniform basis for H, with

$$x = \sum_{k=1}^{m} \tilde{x}_k \tilde{\phi}_k + \sum_{j=-\infty}^{\infty} x_j \phi_j, \tag{6.3}$$

$$g = \sum_{k=1}^{m} \tilde{g}_k \tilde{\phi}_k + \sum_{j=-\infty}^{\infty} g_j \phi_j, \tag{6.4}$$

we have the equivalent system in ℓ^2

$$\frac{d\tilde{x}_k}{dt} = \sigma_k \tilde{x}_a + \tilde{g}_k u, \; k = 1,2,\ldots, m, \quad \frac{dx_j}{dt} = \lambda_j x_j + g_j u,$$
$$-\infty < j < \infty . \tag{6.5}$$

We have seen in Sections 4 and 5 that a scalar n-th order neutral system

(5.1), (5.2), (5.3) has the form (5.4). In (6.5) the fact that $g \in H$ gives

$$\sum_{j=-\infty}^{\infty} |g_j|^2 < \infty$$

for the control distribution coefficients while we have the result (5.15), (5.16) for the control distribution coefficients in (5.4). We intend to argue that the systems (5.1), (5.2), (5.3) constitute the control canonical forms, in the sense of [20], for systems (6.1) of augmented hyperbolic type with eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_m$, λ_j, $-\infty < j < \infty$, satisfying (6.2). The first step in doing so, however, is to establish that every set of complex numbers of this description can be realized as the set of eigenvalues for a system (5.1), (5.2), (5.3) by appropriate choice of $b_0, b_1, \ldots, b_m$ and $\hat{v} \in L^2[-1,1]$ in (5.3). This is the content of

__Theorem 6.1__ __Let__ $\sigma_1, \sigma_2, \ldots, \sigma_m$, λ_j, $-\infty < j < \infty$, __be distinct, the__ λ_j __satisfying an asymptotic relationship__ (6.2), __otherwise arbitrary. Then there are complex numbers__ $b_0, b_1, \ldots, b_{m-1}, b_m$ __and a function__ $\hat{v} \in L^2[-1,1]$ __such that the eigenvalues of the operator__ A __associated with the system__ (5.1), (5.2), (5.3) __are precisely these numbers.__

__Proof__ We first of all determine b_m in (5.3) so that $\alpha = \frac{1}{2}\log(-b_m)$, in agreement with (6.2). We let $\tilde{q}_k$, $k = 1,2,\ldots, m$, q_j, $-\infty < j < \infty$, be the dual basis in $H^m[-1,1]$ relative to the $e^{\sigma_k t}$, $e^{\lambda_j t}/p(\lambda_j)$, as in the preceding section. Then we compute

$$\lambda_j^m e^{\lambda_j}/p(\lambda_j) + b_m \lambda_j^m e^{-\lambda_j}/p(\lambda_j)$$

$$= \frac{\lambda_j^m}{p(\lambda_j)} e^{\lambda_j} \left[1 + b_m e^{-2\lambda_j}\right]$$

$$= \frac{\lambda_j^m}{p(\lambda_j)} e^{\lambda_j} \left[1 + b_m e^{-\log(-b_m)-2j\pi i-2\varepsilon_j}\right]$$

$$= \frac{\lambda_j^m}{p(\lambda_j)} e^{\lambda_j} \left[1 - e^{-2\varepsilon_j}\right] \equiv d_j$$

where, since $\lim\limits_{|j|\to\infty} \frac{\lambda_j^m}{p(\lambda_j)} = 1$, $Re(\lambda_j)$ is bounded, and the ε_j are square summable,

$$\sum_{j=-\infty}^{\infty} |d_j|^2 < \infty .$$

We similarly let

$$\sigma_k^m e^{\sigma_k} + b_m \sigma_k^m e^{-\sigma_k} = \tilde{d}_k, \ k = 1,2\ldots, m.$$

From section 3 the $e^{\sigma_k s}$, $k = 1,2,\ldots, m$, $e^{\lambda_j s}/p(\lambda_j)$, $-\infty < j < \infty$, form a uniform basis for $H^m[-1,1]$ and there exists (see [22], e.g.) a dual biorthogonal basis $\tilde{q}_k$, $k = 1,2,\ldots, m$, q_j, $-\infty < j < \infty$. These also constitute a uniform basis for $H^m[-1,1]$ and we may form the convergent series in $H^m[-1,1]$:

$$d = \sum_{k=1}^{m} \bar{\tilde{d}}_k \tilde{q}_k + \sum_{j=-\infty}^{\infty} \bar{d}_j q_j .$$

If we consider a function

$$w(t) = \sum_{k=1}^{m} \tilde{w}_k e^{\sigma_k t} + \sum_{j=-N}^{N} w_j e^{\lambda_j t} \tag{6.6}$$

where N is an arbitrary positive integer, then $w(t+s)$, $-1 \leq s \leq 1$, is in $H^m[-1,1]$ for any real t. From the biorthogonality of the $\tilde{q}_k$, q_j with respect to the $e^{\sigma_k t}$, $e^{\lambda_j t}$

$$(w(t+\cdot), d)_m = \sum_{k=1}^{m} \tilde{d}_k e^{\sigma_k t} + \sum_{j=-N}^{N} d_j e^{\lambda_j t}$$

$$= \sum_{k=1}^{m} \tilde{w}_k \left(\sigma_k^m e^{\sigma_k} + b_m \sigma_k^m e^{-\sigma_k}\right) e^{\sigma_k t}$$

$$+ \sum_{j=-N}^{N} w_j \left(\lambda_j^m e^{\lambda_j}/p(\lambda_j) + b_m \lambda_j^m e^{-\lambda_j}/p(\lambda_j)\right) e^{\lambda_j t}$$

$$= w^{(m)}(t+1) + b_m w^{(m)}(t-1). \tag{6.7}$$

Taking account of the form of the inner product $(\ ,\ )_m$, (6.7) reads, with $d = \left(d(0), \ldots, d^{(m-1)}(0), d^{(m)}(.)\right)$,

$$w^{(m)}(t+1) + b_m w^{(m)}(t-1) - \sum_{k=0}^{m-1} \overline{d^{(k)}}(0)\, w^{(k)}(t)$$

$$- \int_{-1}^{1} \overline{d^{(m)}(s)}\, w^{(m)}(t+s)\, ds = 0 . \tag{6.8}$$

The system (6.8) is equivalent to one of the form (5.1), (5.2), (5.3) and, since (6.6) is a solution for any N and any choice of the w_k, $\tilde{w}_j$, we conclude that the associated operator A (cf. (4.8)ff.) has eigenvalues $\sigma_1, \ldots, \sigma_m$, λ_j, $-\infty < j < \infty$. Choosing

$$b_k = \overline{d^{(k)}(0)}, \quad \hat{v}(s) = \overline{d^m(s)}$$

then gives the result stated in the theorem.

<u>Corollary 6.2</u> <u>The "spectral fitting" result of Theorem 6.1 remains true for neutral system of negative order</u> $-m$, $m = 1,2,3,\ldots$.

<u>Proof</u> Let the selected spectral values be denoted as λ_j, $j \in J_m$, where J_m is a subset of $\{j \mid -\infty < j < \infty\}$ which is obtained by omitting m integers from that set. We assume the λ_j have the form (6.2). We adjoin to these m numbers $\sigma_1, \sigma_2, \ldots, \sigma_m$. By Theorem 6.1 with $m = 0$, there is a system (5.1), (5.2), (5.3) with these eigenvalues, and we may form the corresponding function (5.17), again with $m = 0$:

$$\hat{\psi}(\omega) = e^{i\omega} + b_0 e^{-i\omega} + \int_{-1}^{1} e^{i\omega s} \hat{v}(s)\, ds . \tag{6.9}$$

The entire function $\hat{\psi}(\omega)$ has the zeros $\sigma_1, \sigma_2, \ldots, \sigma_j$, $j \in J_m$. Then, with $p(\lambda) = (\lambda - \sigma_1)(\lambda - \sigma_2)\ldots(\lambda - \sigma_m)$,

$$\phi(\omega) = \hat{\psi}(\omega)/p(\lambda) \tag{6.10}$$

has precisely the zeros λ_j, $j \in J_m$. Let

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-it\omega} \phi(\omega)\, d\omega,$$

integrated over the real ω axis. Then from (6.9), (.10), $f(t)$ is in $H^{m-1}(R^1)$

with support in $[-1,1]$. From the form of (6.9), $f(t)$ satisfies, in the sense of distribution theory

$$\prod_{k=1}^{m} (D - \sigma_k I) f = \delta_1 + b_m \delta_{-1} + \hat{v}, \quad \hat{v} \in L^2[-1,1],$$

where δ_1, δ_{-1} denote the Dirac distributions with support at $1, -1$, respectively. Thus $f \in H^m[-1,1]$ with non-zero jumps for $f^{(m-1)}$ at $1, -1$. Since $f^{(m-1)}(t)$ for t outside $[-1,1]$, we conclude $f^{(m-1)}(-1+)$ and $f^{(m)}(1-)$ are non-zero. The neutral equation of order $-m$,

$$\int_{-1}^{1} \overline{v(-s)}\, \zeta(t+s)\, ds = \langle \zeta, v \rangle = 0, \quad v(s) = \overline{f(-s)}$$

then has exponential solutions $e^{\lambda_j t}$, $j \in J_m$, and the corollary is proved.

The main result which we wish to establish concerns spectral assignment in augmented or deficient hyperbolic systems by means of linear continuous state feedback. Again we will treat only the case of simple eigenvalues here, leaving a discussion of the multiple case to Section 7. We begin with the augmented case, supposing that we have a system (6.1) with simple eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_k, \lambda_j, -\infty < j < \infty$, the λ_j satisfying (6.2) and the corresponding eigenvectors $\tilde{\phi}_k$, ϕ_j forming a uniform basis for H. Continuous linear state feedback means that we synthesize the control u by a relation

$$u(t) = (x,(t), f)_H, \tag{6.11}$$

for some feedback element $f \in H$. This gives the new "closed loop" system

$$\dot{x} = (A + g \otimes f)x, \quad (g \otimes f)x = (x,f)_H\, g. \tag{6.12}$$

Suppose we specify another set of complex numbers $\rho_1, \rho_2, \ldots, \rho_m, \tau_j$, $-\infty < j < \infty$, as the desired spectrum of the closed loop system. Under what circumstances can this sequence of complex numbers actually be realized as the spectrum of (6.12)?

Theorem 6.3 Let the control distribution coefficients $\tilde{g}_k$, $k = 1,2,\ldots, m$, g_j $-\infty < j < \infty$ for the system (6.1) all be nonzero. By appropriate choice of the feedback element $f \in H$ one may realize as eigenvalues of the closed loop system (6.12) any collection of (distinct, in this theorem) complex numbers $\rho_1, \rho_2, \ldots, \rho_m, \tau_j$, $-\infty < j < \infty$ for which

$$\sum_{j=-\infty}^{\infty} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty \tag{6.13}$$

Proof From Theorem 6.1 we can find a system (5.1), (5.2), (5.3) such that the associated operator A

$$A(z(0), z'(0), \ldots, z^{(m-1)}(0), z^m(.)) = (z'(0), z''(0), \ldots, z^{(m)}(0), z^{(m+1)}(.)) \tag{6.14}$$

defined on the domain in $H^{m+1}[-1,1]$ consisting of z for which

$$z^{(m)}(1) + b_m z^{(m)}(-1) + \sum_{k=0}^{m-1} b_k z^{(k)}(0) + \int_{-1}^{1} z^{(m)}(s)\hat{v}(s)\, ds = 0 \tag{6.15}$$

has precisely the eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_m, \lambda_j$, $-\infty < j < \infty$, of the operator A in (6.1). Consider the transformation $T_0 : H^m[-1,1] \to H$ defined initially by

$$T_0(e^{\sigma_k s}) = \tilde{\phi}_k,\ k = 1,2,\ldots, m,\ T_0(e^{\lambda_j s}/p(\lambda_j)) = \phi_j, -\infty < j < \infty.$$

Since the $e^{\sigma_k s}$, $e^{\lambda_j s}/p(\lambda_j)$ form a uniform basis for $H^m[-1,1]$ while the $\tilde{\phi}_k$, $\tilde{\phi}_j$ form a uniform basis for H, T_0 may be extended by linearity and continuity to a transformation

$$T_0 : H^m[-1,1] \to H$$

which is bounded and has a bounded inverse. Setting

$$x = T_0 y,$$

(6.1) is carried into

$$\dot{y} = T_0^{-1} A T_0 z + T_0^{-1} gu = \mathcal{A} y + T_0^{-1} gu \ . \tag{6.16}$$

The system (6.16) has much the same form as (5.1), (5.2), (5.3) except for the control term. With g given by (6.4),

$$T_0^{-1} g \equiv g(s) = \sum_{k=1}^{m} \tilde{g}_k e^{\sigma_k s} + \sum_{j=-\infty}^{\infty} g_j e^{\lambda_j s}/p(\lambda_j)$$

is convergent in $H^m[-1,1]$ and has a representation

$$T_0^{-1} g = (g(0), g^1(0), \ldots, g^{(m-1)}(0), g^{(m)}(.)).$$

Taking account of the form of $\mathcal{A}$ as shown in (6.14), (6.16) is

$$\frac{\partial y^{(m)}(t,s)}{\partial t} = \frac{\partial y^{(m)}(t,s)}{\partial s} + g^{(m)}(s)u(t) \tag{6.17}$$

$$\frac{dy^{(k)}(t,0)}{dt} = y^{(k+1)}(t,0) + g^{(k)}(0)u(t),\ k = 0,1,\dots, m-1. \quad (6.18)$$

together with the homogeneous boundary condition

$$y^{(m)}(t,1) + b_m y^{(m)}(t,-1) + \sum_{k=0}^{m-1} b_k y^{(k)}(t,0) + \int_{-1}^{1} y^{(m)}(t,s)\hat{v}(s)ds = 0. \quad (6.19)$$

In this process the feedback relation (6.11) transforms via

$$u(t) = (x(t),f)_H = (T_0 y(t),f)_H = (y,(t),\ T_0^* f)_{H^m[-1,1]} . \quad (6.20)$$

The element $f \in H$ is expressed in terms of the dual basis $\tilde{\psi}_k$, $k = 1,2,\dots, m$, $\tilde{\psi}_j$, $-\infty < j < m$, biorthogonal to the eigenvectors $\tilde{\phi}_k$, $k = 1,2,\dots, m$, ϕ_j, $-\infty < j \quad \infty$, of A:

$$f = \sum_{k=1}^{m} \tilde{f}_k \tilde{\psi}_k + \sum_{j=-\infty}^{\infty} f_j \tilde{\psi}_j .$$

Letting $\tilde{q}_k$, $k = 1,2,\dots, m$, q_j, $-\infty < j < \infty$ be the dual basis for $H^m[-1,1]$ biorthogonal to the $e^{\sigma_k s}$, $e^{\lambda_j s}/p(\lambda_j)$, we see easily that $T_0^* \tilde{\psi}_k = \tilde{q}_k$, $T_0^* \tilde{\psi}_j = q_j$, so that

$$T_0^* f \equiv \hat{f}(s) = \sum_{k=1}^{m} f_k q_k(s) + \sum_{j=-\infty}^{\infty} f_j q_j(s). \quad (6.21)$$

At this point the whole problem of spectral assignment has been transferred from (6.1), (6.11) to the system (6.17) - (6.20), since the boundedness and bounded invertibility of T_0 guarantees that the eigenvalues of these two closed loop systems will be identical.

Next we construct a transformation which carried (6.17) - (6.19) into a system of the form (5.1) - (5.3). We have seen in Section 5 that the control distribution coefficients for (5.1) - (5.3) are certain complex numbers $\tilde{b}_k$, $k = 1,2,\ldots, m$, b_j $-\infty < j < \infty$, which are bounded and bounded away from zero. We define $T_1 : H^m[-1,1] \to H$ by

$$T_1(\tilde{b}_k e^{\sigma_k s}) = \tilde{g}_k e^{\sigma_k s}, \; k = 1,w,\ldots m, \tag{6.22}$$

$$T_1(b_j e^{\lambda_j s}/p(\lambda_j)) = g_j e^{\lambda_j s}/p(\lambda_j), \; j = 1,2,\ldots, m \tag{6.23}$$

Since the $\tilde{g}_k$, g_j have been assumed non-zero, T_1 is one to one and has an inverse T_1^{-1} defined on its range. However, the fact that the g_j are square summable shows that T_1^{-1} is unbounded.

We transform (6.17) - (6.19) (equivalently, (6.16)) by setting

$$y = T_1 z. \tag{6.24}$$

Since T_1 commutes with A, (6.16) transforms to

$$\dot{z} = Az + T_1^{-1}T_0^{-1}gu .$$

But (6.22), (6.23) shows that

$$T_1^{-1}T_0^{-1}g = \sum_{k=1}^{m} \tilde{b}_k e^{\sigma_k s} + \sum_{j=-\infty}^{\infty} b_j e^{\lambda_j s}/p(\lambda_j),$$

so that the control distribution coefficients are the same as those for (5.1), (5.2), (5.3). We conclude that (6.24) carries (6.17), (6.18), (6.19) over to (5.1), (5.2), (5.3). The rather formal argument given here can be justified using the mechanism introduced in Section 2 for studying systems whose control distribution elements are not in H.

The feedback relation (6.20), (6.21) transforms to

$$u(t) = (y(t), T_0^* f)_{H^m[-1,1]} = (z,(t), T_1^* T_0^* f)_{H^m[-1,1]} \tag{6.25}$$

and

$$T_1^* T_0^* f \equiv F(s) = \sum_{k=1}^{m} \frac{\tilde{g}_k \tilde{f}_k}{\tilde{b}_k} \tilde{q}_k(s) + \sum_{j=-\infty}^{\infty} \frac{g_j f_j}{b_j} q_j(s)$$

$$\equiv \sum_{k=1}^{m} \tilde{F}_k \tilde{q}_k(s) + \sum_{j=-\infty}^{\infty} F_j q_j(s) \tag{6.26}$$

It will be observed that $\sum_{j=-1}^{\infty} |f_j|^2 < \infty$ together with (5.15), (5.16) implies

$$\sum_{j=-\infty}^{\infty} |F_j/g_j|^2 < \infty \tag{6.27}$$

The essential part of the proof lies in showing that as we pass from the system (5.1), (5.2), (5.3) (with $u(t) \equiv 0$) to the closed loop system (5.1), (5.2), (5.3), (6.25), we may , in so doing, realize any eigenvalues $\rho_1, \rho_2, \ldots, \rho_m$, τ_j, $-\infty < j < \infty$, for the closed loop system which satisfy (6.13). This is carried out for the m-th order case in much the same way as the proof has already been given in [20] for the case $m = 0$.

From (6.13) and Theorem 6.1 there is a system of the form (5.1), (5.2), (5.3) (with $u(t) = 0$) which has the eigenvalues $\rho_1, \rho_2, \ldots, \rho_m$, τ_j, $-\infty < j < \infty$. It has the form

$$\frac{\partial z^{(m)}(t,s)}{\partial t} = \frac{\partial z^{(m)}(t,s)}{\partial s}, \; -1 < s < 1, \; -\infty < t < \infty, \tag{6.28}$$

$$\frac{dz^{(k)}(t,0)}{dt} = z^{(k+1)}(t,0), \; k = 0,1,\ldots, m-1, \tag{6.29}$$

$$z^{(m)}(t,1) + b_m z^{(m)}(t,-1) + \sum_{k=0}^{m-1} a_k z^{(k)}(t,0) + \int_{-1}^{1} z^{(m)}(t,s)\tilde{v}(s)ds = 0 \tag{6.30}$$

for certain complex numbers $a_o, a_1, \ldots, a_{m-1}$ and $\tilde{v} \in L^2[-1,1]$. Let us designate elements of $H^m[-1,1]$:

$$A = (\bar{a}_o, \bar{a}_1, \ldots, \bar{a}_{m-1}, \overline{\tilde{v}(.)}),$$

$$F(\text{cf. } (6.26)) = (F(0), F'(0), \ldots F^{(m-1)}(0), F^{(m)}(.)),$$

$$B = (\bar{b}_0, \bar{b}_1, \ldots, \overline{b_{m-1}}, \overline{\hat{v}(.)}).$$

Then (5.3) has the form

$$z^{(m)}(t,1) + b_m z^{(m)}(t,-1) + (z,B)_m = u(t) \tag{6.31}$$

and use of (6.25). i.e., $u = (z,F)_m$, carries (5.1), (5.2), (5.3) into (6.28), (6.29), (6.30), the last of which reads

$$z^{(m)}(t,1) + b_m z^{(m)}(t,-1) + (z,A)_m = 0 \tag{6.32}$$

just in case

$$B - F = A, \text{ i.e. } F = B - A. \tag{6.33}$$

The question, then, is whether (6.33) can be achieved with F restricted by (6.27). We must, therefore estimate the coefficients of $B - A$ in the expansion

of that element with respect to the uniform basis $\tilde{q}_k$, q_j of biorthogonal elements to the $e^{\sigma_k s}$, $e^{\lambda_j s}/p(\lambda_j)$. Let the relevant coefficients of B and A be $\tilde{B}_k$, B_j and $\tilde{A}_k$, A_j, respectively, i.e.,

$$B = \sum_{k=1}^{m} \tilde{B}_k \tilde{q}_k + \sum_{j=-\infty}^{\infty} B_j q_j, \quad A = \sum_{k=1}^{m} \tilde{A}_k \tilde{q}_k + \sum_{j=-\infty}^{\infty} A_j q_j.$$

We consider also the uniform basis for $H^m[-1,1]$ consisting of the functions $e^{\rho_k s}$, $k = 1,2,\ldots, m$, $e^{\tau_j s}/P(\tau_j)$, $-\infty \quad j \quad \infty$, and we let $\tilde{r}_k$, $k = 1,2,\ldots, m$, r_j, $-\infty < j < \infty$, be the corresponding dual basis of biorthogonal functions. Here (cf. $p(\ ) = \prod_{k=1}^{m} (\lambda - \sigma_k)$)

$$P(\lambda) = \prod_{k=1}^{m} (\lambda - \rho_k).$$

The main result we need is

<u>Lemma 6.4</u> <u>Let</u> $\zeta \in H^m[-1,1]$ <u>have expansions</u>

$$\zeta = \sum_{k=1}^{m} \tilde{\zeta}_k \tilde{q}_k + \sum_{j=-\infty}^{\infty} \zeta_j q_j,$$

$$\zeta = \sum_{k=1}^{m} \tilde{\eta}_k \tilde{r}_k + \sum_{j=-\infty}^{\infty} \eta_j r_j.$$

<u>If the</u> λ_j, τ_j <u>satisfy</u> (6.13), <u>then</u>

$$\sum_{j=-\infty}^{\infty} \left| \frac{\zeta_j - \eta_j}{g_j} \right|^2 < \infty . \qquad (6.34)$$

Proof We define L_j to be the straight line segment joining λ_j to τ_j in the complex plane. For $\lambda \epsilon L_j$ we let

$$P_j(\lambda) = \prod_{k=1}^{m} \left(\lambda - \left[\frac{\lambda - \lambda_j}{\tau_j - \lambda_j} \rho_k + \frac{\tau_j - \lambda}{\tau_j - \lambda_j} \sigma_k \right] \right)$$

if $\tau_j \neq \lambda_j$, otherwise $P_j(\lambda) = p(\lambda) = P(\lambda)$. Then $P_j(\lambda_j) = p(\lambda_j)$, $P_j(\tau_j) =$ $p(\lambda) = P(\lambda)$. Then $P_j(\lambda_j) = p(\lambda_j)$, $P_j(\tau_j) = P(\tau_j)$ and

$$\bar{\zeta}_j - \bar{\eta}_j = (e^{\lambda_j s}/p(\lambda_j), \zeta)_m - (e^{\tau_j s}/P(\tau_j), \zeta)_m$$

$$= \sum_{k=o}^{m-1} \int_{\tau_j}^{\lambda_j} \frac{d}{dx} \left(\frac{\lambda^k}{P_j(\lambda)} \right) \overline{\zeta^{(k)}(0)}\, d\lambda$$

$$+ \int_{\tau_j}^{\lambda_j} \int_{-1}^{1} \frac{d}{d\lambda} \left(\frac{\lambda^m e^{\lambda s}}{P_j(\lambda)} \right) \overline{\zeta^{(m)}(s)}\, ds d\lambda .$$

For $|j|$ sufficiently large the functions $\lambda^k e^{\lambda s}/P_j(\lambda)$ are uniformly bounded for λ in a neighborhood of L_j which is independent of j. Applying the Cauchy estimate for the derivative and performing the indicated integrations we conclude that there is a constant K such that

$$|\bar{\zeta}_j - \bar{\eta}_j| \leq K|\lambda_j - \tau_j|$$

so that (6.13) implies (6.34) and the proof of the lemma is complete.

Now we return to the proof of Theorem 6.2. Let $\tilde{\alpha}_k$, α_j be the expansion coefficients of A with respect to the $\tilde{r}_k$, r_j, i.e.,

$$A = \sum_{k=1}^{m} \tilde{\alpha}_k \tilde{r}_k + \sum_{j=-\infty}^{\infty} \alpha_j r_j .$$

Then

$$\bar{\alpha}_j = (e^{\tau_j s}/P(\tau_j),\ A)_m. \qquad (6.35)$$

But, since functions $z(t,s) = e^{\tau_j(t+s)}$ satisfy (6.32), we have

$$\bar{\alpha}_j = (e^{\tau_j s}/P(\tau_j), A)_m = -\tau_j^m e^{\tau_j}/P(\tau_j) - b_m \tau_j^m e^{-\tau_j}/P(\tau_j).$$

Arguing the same way with the B_j and (6.31)

$$|\bar{\alpha}_j - \bar{B}_j| = |(e^{\tau_j s}/P(\tau_j),A)_m - (e^{\lambda_j s}/p(\lambda_j),A)_m|$$

$$= |\lambda_j^m e^{\lambda_j}/p(\lambda_j) + b_m e^{-\lambda_j}/p(\lambda_j) - \tau_j^m e^{\tau_j}/P(\tau_j) - b_m e^{-\tau_j}/P(\tau_j)|$$

$$= \left| \int_{\tau_j}^{\lambda_j} \frac{d}{d\lambda} \left| \frac{\lambda^m e^{\lambda} + b_m \lambda^m e^{-\lambda}}{P_j(\lambda)} \right| d\lambda \right| \leq \hat{K}|\lambda_j - \tau_j| \qquad (6.36)$$

using the same argument as used in Lemma 6.3. Then, using (6.36) with Lemma 6.3 (with $\zeta = A$)

$$|A_j - B_j| \leq |A_j - \alpha_j| + |\alpha_j - B_j| \leq (K + \hat{K})|\lambda_j - \tau_j|$$

and we conclude that if (6.13) is true and F is given by (6.33), then

$$\left|\frac{F_j}{g_j}\right| = \left|\frac{B_j - A}{g_j}\right| \leq (K + \hat{K}) \left|\frac{\lambda_j - \tau_j}{g_j}\right|$$

so that (6.27) is satisfied. It follows that the feedback (6.25) carrying the system (6.31) into (6.32) can be realized with F satisfying (6.27) if (6.13) is true.

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CLOSED-LOOP EIGENVALUE SPECIFICATION FOR INFINITE DIMENSIONAL S--ETC(U)

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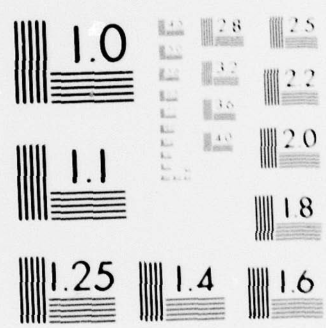

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To complete the proof it is necessary to show that the closed loop system (6.17), (6.18), (6.19), (6.20) with the coefficients appearing in (6.21) determined by

$$\tilde{B}_k - \tilde{A}_k = \tilde{F}_k = \frac{\tilde{f}_k \tilde{g}_k}{\tilde{b}_k}, \quad B_j - A_j = F_j = \frac{f_j g_j}{b_j},$$

$$k = 1,2,\ldots, m, \; -\infty < j < \infty$$

has the same eigenvalues as the system (6.32), i.e., ρ_k, $k = 1,2,\ldots, m$ τ_j, $-\infty < j < \infty$. This does not follow immediately from the fact that (6.24) carries (6.17), (6.18), (6.19), (6.20) into (6.32) when (6.33) obtains because T_1^{-1} is not a bounded operator. In [20] this point was overlooked in the argument which in any case pertained only to the case $m = 0$, so our work here also completes the argument of that paper.

Let the generator of the semigroup associated with (6.35) be $\hat{A}$. Its eigenvalues are ρ_k, $k = 1,2,\ldots, m$, τ_j, $-\infty < j < \infty$, and the corresponding eigenfunctions are $e^{\rho_k s}$, $e^{\tau_j s}/P(\tau_j)$. Let T_2 be defined by (cf. (6.22), (6.23))

$$T_2(\tilde{b}_k e^{\rho_k s}) = \tilde{g}_k e^{\rho_k s}, \tag{6.37}$$

$$T_2(b_j e^{\tau_j s}/P(\tau_j)) = g_j e^{\tau_j s}/P(\tau_j) \tag{6.38}$$

and then be extended to $H^m[-1,1]$ by linearity and continuity. Clearly T_2 commutes with $\hat{A}$, i.e.,

$$T_2\hat{A} = \hat{A}T_2, \quad T_2\hat{A}T_2^{-1} = \hat{A}. \tag{6.39}$$

Let T_1 be the operator defined by (6.22), (6.23). Since (6.24) carries the closed loop system (6.17), (6.18), (6.19), (6.20) into (6.32), we have for the generator $A + g\otimes f$ associated with (6.17), (6.18), (6.19), (6.20),

$$(A + g\otimes f)T_1 = T_1\hat{A}.$$

Multiplying on the right by T_2^{-1} gives

$$(A + g\otimes f)T_1T_2^{-1} = T_1\hat{A}T_2^{-1} = T_1T_2^{-1} = T_1T_2^{-1}T_2\hat{A}T_2^{-1}$$

$$= \text{(from (6.39))} = T_1T_2^{-1}\hat{A} .$$

Thus $T_1T_2^{-1} \equiv T$ has the property

$$(A + g\otimes f)T = T\hat{A}$$

and we conclude that $A + g\otimes f$ has the eigenvalues $\rho_1, \rho_2, \ldots, \rho_m, \tau_j$, $-\infty < j < \infty$, of $\hat{A}$ if we can establish that T is bounded and has a bounded inverse.

Let $\tilde{q}_k$, q_j and $\tilde{r}_k, r_j$, designate the dual bases for $H^m[-1,1]$ biorthogonal to $e^{\sigma_k s}$, $e^{\lambda_j s}/p(\lambda_j)$ and $e^{\rho_k s}$, $e^{\tau_j s}/P(\tau_1)$, respectively. Then T_1 and T_2 have adjoint operators defined by (cf. (6.22),(6.23),

$$T_1^*(\bar{\tilde{b}}_k\tilde{q}_k) = \bar{\tilde{q}}_k\tilde{q}_k,\ T_1^*(\bar{b}_jq_j) = \bar{g}_jq_j,$$

$$T_2^*(\tilde{b}_k\tilde{r}_k) = \tilde{g}_k\tilde{r}_k,\ T_2^*(\bar{b}_jr_j) = \bar{g}_jr_j.$$

Then, at least formally

$$T^* = (T_1 T_2^{-1})^* = (T_2^*)^{-1} T_1^* .$$

The problem is to show this operator is well defined--that the domain of $(T_2^*)^{-1}$ is included in the range of T_1^*. Fortunately this is not difficult. Since the b_j are bounded and bounded away from zero, the range of T_1^* is precisely the set of elements

$$\zeta = \sum_{k=1}^{m} \tilde{\zeta}_k \tilde{q}_k + \sum_{j=-\infty}^{\infty} \zeta_j q_j$$

for which

$$\sum_{j=-\infty}^{\infty} |\zeta_j/g_j|^2 < \infty$$

The domain of $(T_2^*)^{-1}$ is the set of

$$\zeta = \sum_{k=1}^{m} \tilde{\eta}_k \tilde{r}_k + \sum_{j=-\infty}^{\infty} \eta_j r_j$$

for which

$$\sum_{j=-\infty}^{\infty} |\eta_j/g_j|^2 < \infty .$$

Lemma 6.4 shows that these two sets are the same. Thus $(T_2^*)^{-1} T_1^*$ is defined on all of $H^m[-1,1]$ and maps that space in a one to one fashion onto itself. Clearly it is a closed operator and the closed graph theorem implies then that it is bounded and boundedly invertible. The proof of Theorem 6.3 is now complete.

Theorem 6.5 The result of Theorem 6.3 remains valid for deficient hyperbolic systems, (6.13) being replaced by

$$\sum_{j \in J_m} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty. \tag{6.40}$$

(Assuming, as before, that the $g_j \neq 0$.).

The proof is essentially the same as the one already given and will not be repeated in complete detail. A deficient hyperbolic system is first of all mapped, by means of a transformation T_0 (cf. (6.15)ff.) defined by

$$T_0(p(\lambda_j)e^{\lambda_j s}) = \phi_j, \; j \in J_m$$

into a neutral system of negative order $-m$ in $H^m[-1,1]$. Such a system may be represented by

$$\frac{\partial z(t,s)}{\partial t} = \frac{\partial z(t,s)}{\partial s} + g(s)u(t), \; g \in H^{-m}[-1,1], \tag{6.41}$$

or, equivalently, as

$$\dot{z} = Az + gu, \tag{6.42}$$

A being the differentiation operator: $A: \mathcal{D}(A) \subseteq H^{-m+1}[-1,1] \rightarrow H^{-m}[-1,1]$, the domain, $\mathcal{D}(A)$, being the subset of $H^{-m+1}[-1,1]$ defined by

$$\langle z,v \rangle = 0 \tag{6.43}$$

The corresponding canonical system is

$$\frac{\partial y(t,s)}{\partial t} = \frac{\partial y(t,s)}{\partial s} \tag{6.44}$$

or

$$\frac{dy}{dt} = Ay \tag{6.45}$$

with a nonhomogeneous boundary condition

$$(y,v) = u \tag{6.46}$$

Again the transformation T_1 which carries (6.42), (6.43) into (6.45), (6.46) is defined in terms of the respective control distribution coefficients for the two systems, i.e.,

$$T_1(b_j e^{\lambda_j s}) = g_j e^{\lambda_j s}, \; j \in J_m.$$

The g_j are square summable and we have established at the end of Section 5 that the b_j are bounded and bounded away from zero. Thus, as in Theorem 6.3, T_1 is bounded but does not have a bounded inverse.

A feedback relation for the original system

$$u = (x,f)_H$$

is mapped by T_0 into a corresponding feedback law for (6.42), (6.43),

$$u = (z, T_0^* f)_{-m}$$

and then by T_1 into

$$u = (y, T_1^*, T_0^* f)_{H^{-m}[-1,1]} \equiv (y,F)_{-m} \tag{6.47}$$

We may replace this relation by

$$u = \langle y, \tilde{F} \rangle$$

where $\tilde{F}$ is the element of $H_0^m[-1,1]$ which defines the same linear functional on $H^{-m}[-1,1]$ as does $F \in H^{-m}[-1,1]$ via (y,F). The closed loop system is thus

$$\langle y, v - \tilde{F} \rangle = 0 \qquad (6.48)$$

Here $\tilde{F}$ is a "lower order" term by comparison wiht V: we have $v \in H_0^{m-1}[-1,1] \cap H^m[-1,1]$, $v^{(m-1)}(-1) \neq 0$, $v^{(m-1)}(1) \neq 0$, while $\tilde{F} \in H^m[-1,1]$. Given desired eigenvalues

$$\tau_j = \alpha + j\pi i + \nu_j, \; j \in J_m$$

with $\sum_{j \in J_m} |\nu_j|^2 < \infty$, we have seen in Corollary 6.2 that there is an element, call it V, such that the eigenvalues of (6.45) with

$$\langle y, V \rangle = 0 \qquad (6.49)$$

are these τ_j, $j \in J_m$. The proof of Corollary 6.2 shows that the values of $V^{(m-1)}$ at $-1,1$ are, except for a constant multiple, determined by α to be the same as those of $v^{(m-1)}$ at those points. This means (6.49) can be written as

$$\langle y, v - G \rangle = 0 \qquad (6.50)$$

for some $G \in H_0^m[-1,1]$. Then (6.45), (6.48) has the same eigenvalues as (6.45), (6.50) if $\tilde{F} = G$. If $\tilde{F}$ is expanded in terms of the elements ϕ_j biorthogonal

to the $p(\lambda_j)e^{\lambda_j s}$, $j \epsilon J_m$,

$$\tilde{F} = \sum_{j \epsilon J_m} F_j \phi_j,$$

then

$$\sum_{j \epsilon J_m} |F_j/g_j|^2 < \infty,$$

as before. The essential question then reduces the one treated in

<u>Lemma 6.6</u> <u>Suppose the eigenvalues of</u> (6.45),(6.46) <u>are</u>

$$\lambda_j = \alpha + j\pi i + \varepsilon_j, \; j \epsilon J_m, \; \sum_{j \epsilon J_m} |\varepsilon_j|^2 < \infty$$

<u>while those of</u> (6.45), (6.50) <u>are</u> τ_j, $j \epsilon J_m$, <u>with</u>

$$\sum_{j \epsilon J_m} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty .$$

<u>Then if</u>

$$G = \sum_{j \epsilon J_m} G_j \phi_j$$

<u>we have</u>

$$\sum_{j \epsilon J_m} |G_j/g_j|^2 < \infty . \tag{6.51}$$

<u>Proof</u> Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be m complex numbers distinct from the λ_j or τ_j. Let $L = \prod_{k=1}^{m} (D - {}_kI)$. In (6.46) and (6.50) we let $y = Lw$, $w \epsilon L^2[-1,1]$. Then (6.48), (6.50) are replaced by

$$\int_{-1}^{1} Lw(s)(\bar{v}(-s))ds = 0,$$

$$\int_{-1}^{1} Lw(s)(\bar{v}(-s) - \overline{G(-s)})ds - 0,$$

respectively. Repeated integration by parts yields (cf. (6.9))

$$\overline{v^{m-1}}(-1)w(1) + b\,\overline{v^{(m-1)}}(1)w(-1) + \int_{-1}^{1} w(s)\,\overline{\bar{L}v(-s)}ds = 0 \tag{6.45}$$

$$\overline{v^{m-1}}(-1)w(1) + b\overline{v^{(m-1)}}(1)w(-1) + \int_{-1}^{1} w(s)\left|\overline{\bar{L}v(-s) - \bar{L}G(-s)}\right| ds = 0 \tag{6.46}$$

The exponentials which satisfy (6.45) are $e^{\lambda_j s}$, $-\infty < j < \infty$ and those satisfying (6.46) are $e^{\tau_j s}$, $-\infty < j < \infty$, both include $\sigma_1, \sigma_2, \ldots, \sigma_m$ now. Applying the results used to prove Theorem 6.3 in the case $m = 0$, the expansion of $\bar{L}G(-s)$ in terms of the biorthogonals to the $e^{\lambda_j s}$,

$$\bar{L}G(-s) = \sum_{j=-\infty}^{\infty} H_j \phi_j$$

have the property

$$\sum_{j=-\infty}^{\infty} |H_j/g_j|^2 < \infty . \tag{6.52}$$

(Actually, the H_k which corresond to the σ_k are all zero since those eigenvalues are invariant as we pass from (6.45) to (6.46).) Now

$$\bar{H}_j = \int_{-1}^{1} e^{\lambda_j s}\,\overline{\bar{L}G(-s)}ds = (G \in H_0^m[-1,1]) = \int_{-1}^{1} p(\lambda_j)e^{j s}\,\overline{G(-s)}ds = \bar{G}_j .$$

Thus (6.52) gives (6.51) and the proof is complete. The remainder of the proof of Theorem 6.5 is essentially that of Theorem 6.3 and is omitted.

Finally there is the question of what spectral assignment results may be obtained if some of the control distribution coefficients g_j for the system (6.1) are equal to zero. First of all, it is quite clear from the form (6.5)

of the system that whenever $g_j = 0$, the corresponding λ_j (or σ_k if $\tilde{g}_k = 0$) is an eigenvalue which cannot be affected by feedback. If finitely many $g_j = 0$, one may simply exclude linear combinations of the corresponding ϕ_j from the space under consideration, forming a new space $\hat{H}$ from linear combinations of the ϕ_j for which $g_j \neq 0$. The new system

$$\frac{d\hat{x}}{dt} = \hat{A}\hat{x} + \hat{g}u, \ \hat{x}, \ \hat{g} \in \hat{H}$$

will have eigenvalues σ_k, λ_j for which $\tilde{g}_k$, $g_j \neq 0$, only finitely many less than before. Therefore it will be either an augmented or deficient hyperbolic system and the foregoing theory can be applied to show that the σ_k, λ_j corresponding to $\tilde{g}_k$, $g_j \neq 0$ can be changed to new values ρ_k, τ_j, as long as

$$\sum_{j \ni g_j \neq 0} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty$$

If infintely many $g_j = 0$ we cannot pursue this route because the modified system will, in general, not be either an augmented on deficient hyperbolic system. It seems likely in this case that one might successfully proceed in the following manner. Let the index set J be divided into mutually disjoint subsets J_0, J_1 with $J_0 \cup J_1 = J$,

$$g_j = 0, \ j \in J_0, \ g_j \neq 0, \ j \in J_1$$

Let $\{\alpha_j \mid j \in J_0\}$ be square summable with $\alpha_j \neq 0$, $j \in J_0$, and then, for $\varepsilon > 0$, let

$$G_j(\varepsilon) = \begin{cases} g_j, & j \in J_1 \\ \varepsilon\alpha_j, & j \in J_0 \end{cases}.$$

We consider the problem of moving the σ_k, λ_j to new values ρ_k, τ_j with

$$\sigma_k = \rho_k, \ \lambda_j = \tau_j \quad \text{if} \quad k \text{ or } j \in J_0$$

$$\sum_{j \in J_1} \left| \frac{\tau_j - \lambda_j}{g_j} \right|^2 < \infty .$$

As one examines the proof of Theorem 6.3 we see that the corresponding $T_0(\varepsilon)$, $T_1(\varepsilon)$, $T_2(\varepsilon)$ may all be defined and have limits as $\varepsilon \to 0$, but $T_1(0)$, $T_2(0)$ are not invertible. However, it is only necessary to establish that

$$\lim_{\varepsilon \to 0} T(\varepsilon) = T_1(\varepsilon) T_2(\varepsilon)^{-1} = T(0) \tag{6.53}$$

exists, is bounded and boundedly invertible, in order to see that the new eigenvalues can be realized by linear feedback for the system with control distribution coefficients $G_j(0)$--which are of course the original g_j. Establishing the boundedness and bounded invertibility of (6.53) is done in a manner quite similar to the proof of Lemma 6.4, and this is left to the reader.

7. Modifications for Multiple Eigenvalues

We have discussed only the case of simple eigenvalues in the main body of the manuscript. The treatment of multiple eigenvalues, while not conceptually difficult, is a rather detailed business and would have made an already lengthy manuscript too long if we had been forced to consider the problems of multiplicity at every step in the development. Nevertheless, a complete treatment requires attention to the possibility of multiple eigenvalues and, in this final section, we indicate how this is done.

We will assume that we are dealing with a system (1.1) of augmented or deficient hyperbolic type. We admit now the possibility that A may have finitely many eigenvalues, without loss of generality these may be called $\lambda_1, \lambda_2, \ldots, \lambda_r$, which have associated multiplicities $\mu_1, \mu_2, \ldots, \mu_r$ greater than 1. There exist systems having infinitely many multiple eigenvalues, but we will not consider them here. For such systems Theorem 6.3 (or Theorem 6.5) must be replaced by a more complicated result. The systems listed as examples in Section 2 and the neutral systems of positive and negative order studied in Sections 4, 5 and 6 have at most finitely many multiple eigenvalues; in the case of the neutral systems this is a corollary of our application of Rouché's theorem.

Section 1 has already been developed with the possibility of multiplicity taken into account. The first result which is explicitly restricted to the simple case is Theorem 3.1. To begin we need some conventions. As we have already indicated, the multiple eigenvalues will be designated $\lambda_1, \lambda_2, \ldots, \lambda_r$ and the multiplicities as $\mu_1, \mu_2, \ldots, \mu_r$. We will always assume that the multiple eigenvalues are included in the λ_j, $j \in J$, the $\sigma_1, \sigma_2, \ldots, \sigma_m$ will always be simple. This might at first seem to imply some restriction, but

this is not the case. If we begin with a uniform basis of generalized exponentials for $L^2[-1,1]$ which involve exponents $\lambda_1,\lambda_2,\ldots,\lambda_\rho$ with multiplicities $\nu_1,\nu_2,\ldots,\nu_\rho$ and wish to adjoin m generalized exponentials which involve exponents $\lambda_1,\lambda_2,\ldots,\lambda_\rho,\lambda_{\rho+1},\ldots,\lambda_r$ with multiplicities $\hat{\nu}_1,\hat{\nu}_2,\ldots\hat{\nu}_\rho,\hat{\nu}_{\rho+1},\ldots,$ we carry out an exchange. From the λ_j, $j\in J$, we select $\hat{\nu} = \sum_{\ell=1}^{r}\hat{\nu}_\ell$ complex numbers such that the $e^{\lambda_j s}$ occur as simple exponentials in the basis for $L^2[-1,1]$. Call these $\tilde{\lambda}_{k,\ell}$ $k = 1,2,\ldots,r$, $\ell = 1,2,\ldots,\nu_k$. We move the corresponding functions out of the basis for $L^2[-1,1]$, replacing the simple exponentials $e^{\lambda_{k,\ell}s}$ by

$$\frac{s^n}{n!}e^{\lambda_j s},\quad n = \nu_j,\nu_{j+1},\ldots,\nu_j+\hat{\nu}_j-1,\ j = 1,2,\ldots,\rho$$

$$\frac{s^n}{n!}e^{\lambda_j s},\ n = 0,1,\ldots,\hat{\nu}_j-1,\ j = \rho+1,\ldots,r.$$

The $\tilde{\lambda}_{k,\ell}$ are renamed σ with appropriate indices and the $e^{\lambda_{k,\ell}s}$ are combined with the simple exponentials in the set originally to be adjoined. We are then adjoining m simple exponentials $e^{\lambda_k s}$ to a basis for $L^2[-1,1]$ which may include generalized exponentials $\frac{s^n}{n!}e^{\lambda_j s}$, $n = 0,1,\ldots,\mu_j - 1$. Are we justified in assuming that our exchange replaced a uniform basis for $L^2[-1,1]$ by another uniform basis? The answer is yes. From [12], [14], [24] the original basis is associated with an entire function $\psi(\lambda)$ of order 1, type 1, such that

(i) for each λ_j associated with the original basis, $\psi(\lambda)$ has a zero of multiplicity ν_j ($\nu_j = 1$ unless $j = 1,2,\ldots,\rho$) at λ_j;

(ii) $\psi(\lambda)$ does not belong to $L^2(-\infty,\infty)$ on the real axis but $\psi(\lambda)/(1+|\lambda|)$ does lie in this space.

Obviously $\psi(\lambda)$ can be written in the form

$$\psi(\lambda) = \prod_{j=1}^{\rho} (\lambda - \lambda_j)^{\nu_j} \prod_{k=1}^{r} \prod_{\ell=1}^{\hat{\nu}_k} (\lambda - \lambda_{k,\ell}) \phi(\lambda).$$

The exchange process corresponds to replacing $\psi(\lambda)$ by

$$\theta(\lambda) = \prod_{j=1}^{\rho} (\lambda - \lambda_j)^{\nu_j + \hat{\nu}_j} \prod_{j=\rho+1}^{r} (\lambda - \lambda_j)^{\hat{\nu}_j} \phi(\lambda)$$

and it is easily seen that $\theta(\lambda)$ has the same properties as $\psi(\lambda)$ except for the location and character of finitely many zeros. From the properties of $\theta(\lambda)$ and the theory in [12], [14], [24] it is easy to see that the new generalized exponentials are strongly independent and span $L^2[-1,1]$. The argument which appears in Section 3, (3.33).ff., of [19] may then be used to see that we have a uniform basis for $L^2[-1,1]$.

Once the exponentials being adjoined are simple and distinct from those already in the basis for $L^2[-1,1]$, the proof of Theorem 3.1 proceeds essentially the same way as it appears in Section 3 now except for one point. In forming the solution $\hat{z}(s)$ of (3.13) one must now consider solutions of

$$\prod_{k=1}^{m} (D - \sigma_k I)\hat{z} = \ldots + \sum_{\ell=0}^{\mu_j - 1} f_{j,\ell} \frac{s^\ell}{\ell!} e^{\lambda_j s} + \ldots \tag{7.1}$$

Since $e^{\lambda_j s}/p(\lambda_j)$, $p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k)$, solves

$$\prod_{k=1}^{m} (D - \sigma_k I)(e^{\lambda_j s}/p(\lambda_j)) = e^{\lambda_j s}$$

and since

$$\frac{1}{\ell!}\left.\frac{d^\ell e^{\lambda s}}{d\lambda^\ell}\right|_{\lambda=\lambda_j} = \frac{s^\ell}{\ell!}\, e^{\lambda_j s} ,$$

a solution of (7.1) may be found in the form

$$\hat{z}(s) = \ldots + \sum_{\ell=0}^{\mu_j-1} \frac{f_{j,\ell}}{\ell!} \left.\frac{d^\ell (e^{\lambda s}/p(\lambda))}{d\lambda^\ell}\right|_{\lambda=\lambda_j} + \ldots$$

and the indicated terms will have the form

$$\hat{z}(s) = \ldots + \sum_{\ell=0}^{\mu_j-1} F_{j,\ell} \frac{s^\ell}{\ell!} e^{\lambda_j s}/p(\lambda_j) + \ldots$$

The fact that $Lz = 0$ has no solutions of the form $s^\ell/\ell!\, e^{\lambda_j s}$ is easily used to show that the linear relationship between the $f_{j,\ell}$ and $F_{j,\ell}$ is then completed without difficulty and the final result is that $e^{\sigma_1 s}, \ldots, e^{\sigma_m s}$ with the $e^{\lambda_j s}/p(\lambda_j), \left\{\frac{s^\ell}{\ell!} e^{\lambda_j s}\right\}/p(\lambda_j)$, as appropriate, form a uniform basis for $H^m[-1,1]$. In the case of infinitely many multiple λ_j one has to consider the uniform boundedness and uniform invertibility of the matrices relating the $F_{j,\ell}$ to the $f_{j,\ell}$ and this gives rise to some difficulties which fortunately we may ignore here.

In the proof of Theorem 3.2 the $\sigma_1, \ldots, \sigma_m$ corresponding to the excluded $e^{\sigma_1 s}, \ldots, e^{\sigma_m s}$ are taken to be simple and different from the remaining λ_j. The proof then proceeds as before except that when one sets $w = L\zeta$, when we have

$$\zeta(s) = \ldots + \sum_{\ell=0}^{\mu_j-1} \zeta_{j,\ell} \frac{s^\ell}{\ell!} e^{\lambda_j s} + \ldots$$

then

$$L\left(\frac{s^\ell}{\ell!} e^{\lambda_j s}\right) = \frac{1}{\ell!} L \left.\left(\frac{d^\ell e^{\lambda s}}{d}\right)\right|_{\lambda=\lambda_j} = \frac{1}{\ell!} \frac{d^\ell}{d\lambda^\ell} \left.(L\, e^{\lambda s})\right|_{\lambda=\lambda_j} = \frac{1}{\ell!} \frac{d^\ell}{d\lambda^\ell} (p(\lambda) e^{\lambda s})_{\lambda=\lambda_j}$$

is, since $p(\lambda_j) \neq 0$, expressible as a linear combination of $\frac{s^n}{n!} p(\lambda_j)e^{\lambda_j s}$, $n = 0,1,\ldots\ \ell$, the coefficients involving p and its derivatives evaluated at (λ_j). So, with $w = L\zeta$ we have

$$w(s) = \ldots + \sum_{\ell=0}^{\mu_j - 1} z_{j,\ell} \frac{s^\ell}{\ell!} p(\lambda_j)e^{\lambda_j s} + \ldots ,$$

and again the linear relationship between the $\zeta_{j,\ell}$ and the $z_{j,\ell}$ is easily seen to be nonsingular.

In Section 4 we note that as soon as we admit the possibility of multiple eigenvalues these correspond to multiple zeros of the function $\Phi(\lambda)$ defined by (4.10). The proof of Theorem 4.2 is no different, but it does show that the λ_j are simple for sufficiently large $|j|$. A word needs to be said about the meaning of (4.11). Again assuming the $\sigma_1, \sigma_2, \ldots, \sigma_m$ to be simple zeros, each λ_j in (4.11) must be listed a number of times corresponding to its multiplicity: this could be done, e.g., if λ_1 has multiplicity μ_1, by letting $\lambda_1 = \lambda_2 = \ldots = \lambda_{\mu_1}$. Since there are only finitely many non-simple zeros, this does no harm to the asymptotic relationship between λ_j and $\alpha + j\pi i$. These remarks continue to be valid in connection with Theorem 4.4.

The first place in the paper where a really significant difference associated with the control distribution coefficients as discussed in Section 5. We re-index the elements of the dual basis so that corresponding to a λ_j of multiplicity μ_j we have elements $q_{j,\ell}$, $\ell = 1,2,\ldots,\mu_j$ such that

$$\left(\frac{s^{\ell-1}}{(\ell-1)!} e^{\lambda_j s}/p(\lambda_j),\ q_{\hat{j},\hat{\ell}}\right)_m = \begin{cases} 1, & j = \hat{j},\ \ell = \hat{\ell}. \\ 0 & \text{otherwise} \end{cases} \tag{7.2}$$

We do the same for the control distribution coefficients themselves, writing them as b_j^ℓ. In place of (5.14) we have, in the case of multiplicity,

$$\overline{\langle q_{j,\ell}, b\rangle} = \overline{q_{j,\ell}^{(m)}(1)} = b_j^{\ell} \tag{7.3}$$

With solutions $z(t,s)$ of (5.1), (5.2), (5.3) now involving terms

$$z(t,s) = \ldots + \sum_{\ell=1}^{\mu_j} z_{j,}(t) \frac{s^{\ell-1}}{(\ell-1)!} e^{\lambda_j s} + \ldots$$

the fact that

$$\frac{\partial z(t,s)}{\partial t} = \ldots + \sum_{\ell=1}^{\mu_j} \dot{z}_{j,\ell}(t) \frac{s^{\ell-1}}{(\ell-1)!} e^{\lambda_j s} + \ldots$$

$$= \frac{\partial z(t,s)}{\partial s} = \ldots + \sum_{\ell=1}^{\mu_j} z_{j,\ell}(t) \left(\frac{s^{\ell-2}}{(\ell-2)!} e^{\lambda_j s} + j\frac{s^{\ell-1}}{(\ell-1)!} e^{\lambda_j s} \right) + \ldots$$

shows that we will have, in each case of a multiple λ_j,

$$\frac{d}{dt} \begin{pmatrix} z_{j,1}(t) \\ z_{j,2}(t) \\ \vdots \\ z_{j,\mu_j}(t) \end{pmatrix} = \begin{pmatrix} \lambda_j & 1 & 0 \ldots & 0 \\ 0 & \lambda_j & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \lambda_j \end{pmatrix} \begin{pmatrix} z_{j,1}(t) \\ z_{j,2}(t) \\ \vdots \\ z_{j,\mu_j}(t) \end{pmatrix} + \begin{pmatrix} b_j^1 \\ b_j^2 \\ \vdots \\ b_j^{\mu_j} \end{pmatrix} u(t) \tag{7.3}$$

Since every such system with $b_j^{\mu_j} \neq 0$ can be reduced to a system of the form (1.18), and since only finitely many λ_j are assumed to have multiplicity $\mu_j > 1$, all that needs to be established is that $b_j^{\mu_j} \neq 0$ in systems (7.3) arising from (5.1), (5.2), (5.3).

Corresponding to a zero $\omega_j = -i\lambda_j$ of $\hat{\psi}(\omega)$ (cf. (5.17)) of multiplicity μ_j we let

$$\psi_{j,\ell}(\omega) = \frac{i^{\ell} \mu_j!(\lambda_j)\hat{\psi}(\omega)}{\hat{\psi}^{(\mu_j)}(\omega j)(\omega-\omega_j)^{\mu_j-\ell}}, \quad \ell = 0,1,\ldots, \mu_j - 1$$

Since

$$\hat{\psi}(\omega) = \frac{\hat{\psi}^{(\mu_j)}(\omega_j)}{\mu_j!}(\omega - \omega_j)^{\mu_j} + \ldots$$

We have

$$\psi_{j,\ell}(\omega) = i^{\ell}\, p(\lambda_j)(\omega - \omega_j)^{\ell} + \ldots .$$

If we define elements $v_{j,\ell}$ in terms of $\psi_{j,\ell-1}$ in the same way as the v_j were defined in terms of the ψ_j following (5.21), then

$$\psi_{j,\ell-1)}(\omega) = (e^{i\omega s}, v_{j,\ell})_m,\ \ell = 1,2,\ldots,\ \mu_j.$$

For $\lambda_{\hat{j}} = i\omega_{\hat{j}}$ other than $\lambda_j = i\omega_j$ we have

$$\psi_{j,\ -1}(\omega_{\hat{j}}) = (e^{i\omega_{\hat{j}}s}, v_{j,\ell})_m = (e^{\lambda_{\hat{j}}s}, v_{j,\ell})_m = (e^{{}_{j}{}^{s}}, v_{j,})_m = 0 \tag{7.4}$$

If $\lambda_{\hat{j}}$ has multiplicity $\mu_{\hat{j}}$ then $\psi_{j,\ell}^{(k-1)}(\omega_{\hat{j}}) = 0,\ k = 1,2,\ldots,\ \mu_{\hat{j}},$ so that

$$\frac{(-i)^{k-1}}{(k-1)!}\psi_{\hat{j},\ell-1}^{(k-1)}(\omega_{\hat{j}}) = \left(\frac{s^{k-1}}{(k-1)!}e^{\lambda_{\hat{j}}s}, v_{j,\ell}\right)_m = 0. \tag{7.5}$$

For λ_j itself, we note that for $k = 1,2,\ldots,\ \ell-1$,

$$\frac{(-i)^{k-1}}{(k-1)!}\psi_{j,\ell}^{(k-1)}(\omega_j) = \left(\frac{s^{k-1}}{(k-1)!}e^{\lambda_j s}, v_{j,\ell}\right)_m = 0 \tag{7.6}$$

while

$$\frac{(-i)^{\ell-1}}{(\ell-1)!}\,\psi_{j,\ell-1}^{\ell-1}(\omega_j) = \left[\frac{s^{\ell-1}}{(\ell-1)!}e^{\lambda_j s},\ v_{j\ell}\right]_m = p(\lambda_j),\ \ell = 1,2,\ldots,\ \mu_j \tag{7.7}$$

The relationships (7.4) - (7.7) imply that the $v_{j,\ell}$ are related to the biorthogonal elements $q_{j,\ell}$ defined by (7.2) via

$$\begin{aligned}
v_{j,1} &= q_{j,1} + a_{12}q_{j,2} + a_{13}q_{j,3} + \ldots + a_{1\mu_j}q_{j,\mu_j} \\
v_{j,2} &= \qquad\quad q_{j,2} + a_{2,3}q_{j,3} + \ldots + a_{2\mu_j}q_{j,\mu_j} \\
&\ \ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
v_{j,\mu_j} &= \qquad\qquad\qquad\qquad\qquad\qquad\quad q_{j,\mu_j},
\end{aligned}$$

the $a_{\ell,k}$, $k = \ell+1,\ldots,\ \mu_j$ being coefficients which are determined by derivatives of $\psi_{j,\ell}$ of orders $\ell+1,\ldots,\ \mu_j$. For the evaluation of $b_j^{\mu_j}$ we fortuitously have

$$b_j^{\mu_j} = \overline{\langle q_{j,\mu_j},\ b\rangle} = \overline{\langle v_{j,\mu_j},\ b\rangle} = \overline{v_{j,\mu_j}^{(m)}(1)}\ .$$

Since the definition of ψ_{j,μ_j-1} here parallels that of ψ_j in (5.18) except *for* the replacement of $1/\psi'(\omega_j)$ by $i^{\mu_j-1}\mu_j!/\hat{\psi}^{(\mu_j)}(\omega_j)$, the same methods as were used in Section 5 to obtain (5.25) may again be used to evaluate b_{j,μ_j} here and to show that it is not zero. In fact we have

$$b_j^{\mu_j} = \overline{\left(\frac{i^{\mu_j}\mu_j!\,p(\lambda_j)}{\hat{\psi}^{(\mu_j)}(\omega_j)}\right)}\ .$$

Once this result has been established the remaining theorems of Section 6 can be proved, allowing for finitely many multiple λ_j or τ_j, without difficulty just by replacing the q_j or r_j by $q_{j,\ell}$, $r_{j,\ell}$ dual to $(s^{\ell-1}/(\ell-1)!)e^{\lambda_j s}(p(\lambda_j)$ or $(s^{\ell-1}/(\ell-1)!e^{\tau_j s}/P(\tau_j))$. The map T_0 is simply replaced by $PB(\hat{P}\hat{B})^{-1}$ where P and B are as in (1.9), (1.13), reducing the original system to control Jordan form and $\hat{P}\hat{B}$ is the corresponding transformation reducing the related system (6.17), (6.18) (6.19), having the same spectral structure and control distribution coefficients, to the same control Jordan form.

REFERENCES

1. Adams, R.A.: "Sobolev Spaces", Academic Press, New York, San Francisco, London 1975.

2. Bellman, R., and K.L. Cooke: "Differential-Difference Equations", Academic Press, New York 1963.

3. Brunovsky, P.: "A classification of linear controllable systems", Kybernetika 6(1970), pp. 173-188.

4. Courant, R., and D. Hilbert: "Methods of Mathematical Physics, Vol. I", John Wiley and Sons, Inc., New York 1953.

5. Dolecki, S., and D.L. Russell: "A general theory of observation and control", SIAM J. Cont. Opt., Vol. 15 (1977), pp. 185-220.

6. Dunford, N., and J.T. Schwartz: "Linear Operators, Part I: General Theory", Interscience Pub. Co., New York, 1958.

7. Graham, K.D. and D.L. Russell: "Boundary value control of the wave equation in a spherical region", SIAM J. Contr., 13 (1975), pp. 174-196.

8. Graves, L.M.: "Theory of Functions of Real Variables", McGraw-Hill Book Co., New York 1956.

9. Hale, J.: "Functional Differential Equations", Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York 1971.

10. Kato, T.: "Perturbation Theory for Linear Operators", Springer-Verlag, New York 1966.

11. P.D. Lax: "On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic quations", Comm. Pur. App. Math, Vol 8 (1955), pp. 615-633.

12. Levinson, N.: "Gap and density theorems", Amer. Math. Soc Colloq. Publ., 26, Providence, R.I., 1940.

13. Lions, J.L. and E. Magenes: "Problèmes aux limites nonhomogènes et applications", Vol. 1, Dunod, Paris 1968.

14. Paley, R.E.A.C. and N. Wiener: "Fourier Transforms in the Complex Domain", Amer. Math. Soc. Colloq. Publ., 19, Providence, R.I. 1934.

15. P.H. Rabinowitz: "Periodic solutions of nonlinear hyperbolic partial differential equations", Comm. Pur. App. Math., Vol. 20 (1967) pp. 145-204.

16. Reid, R.M.: "Some controllability and stabilizability properties of linear water waves" PhD Thesis, University of Wisconsin, Madison, August 1979.

17. Riesz, F. and B. Sz.-Nagy: "Functional Analysis", Frederick Ungar Pub. Co., New York 1955.

18. Rudin, W.: "Real and Complex Analysis", McGraw-Hill Book Co., New York 1966.

19. Russell, D.L.: "Nonharmonic Fourier series in the control theory of distributed parameter systems", J. Math. Anal. Appl., 18 (1967) pp. 542-559.

20. Russell, D.L." "Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems", J. Math. Anal. Appl. Vol. 62 (1978) pp. 186-225.

21. Russell, D.L.: "Differential-delay equations as canonical forms for controlled hyperbolic systems with applications to spectral assignment", in "Control Theory of Systems Governed by Partial Differential Equations, A.K. Aziz, J.W. Wingate, M.J. Balas, Eds., Academic Press, N.Y. 1977.

22. Russell, D.L.: "Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions", SIAM Review, Vol. 20 (1978) pp. 639-789.

23. Russell, D.L.: "Mathematics of Finite Dimensional Control Systems", Marcel Dekker, Inc., New York 1979.

24. Schwartz,L.: "Étude des sommes d'exponentielles", deuxième édition, Hermann, Paris 1950.

25. Triggiani, R.: "Extensions of rank conditions for controllability and observability to Banach spaces and unbounded operators", SIAM J. on Control, 14 (1976) pp. 313-338.

26. Ullrich, D.: "Difference quotients and systems of nonharmonic Fourier series", to appear.

27. Watson, G.N.: "A treatise on the Theory of Bessel Functions",

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We introduce a method for reduction of a class of distributed parameter control systems to a corresponding control canonical form which has the same intermediate "control Jordan form" as the original system. Using this reduction together with certain results concerning the completeness and independence properties of complex exponentials we are able to study the problem of eigenvalue specification (pole placement for certain classes of hyperbolic systems which are more general than those discussed in the earlier paper [20].

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